Detection of Irregular Assignments of Cases to Judges

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Abstract

I develop tools to detect irregular assignments of cases to judges and apply them to Ecuador's judicial system. I derive the sharp bounds on the overall, court-specific, and judge-specific probabilities that a case's assignment is inconsistent with existing regulations. The bounds rely on administrative case assignment data and one, or both, of the following assumptions: (i) that certain observed case characteristics do not influence which judge a case should be assigned to, and (ii) that the probability distribution over the judges that each case should be assigned to is known (e.g. uniform, random assignment). I construct a database of all publicly-available case assignments in Ecuador's district courts, with over two million assignments from 2016 to 2020 and I find that 5% of courts and judges account for 43% and 37% of irregular assignments, respectively. Overall, at least 65 thousand assignments, 2.9%, are irregular.

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Regulations that govern the assignment of judicial cases to judges are ubiquitous and can be traced back to the 4th century BCE in Athens.[1] They aim to abolish the market for judges, where judicial decisions tend to favor the party with a higher willingness to pay for a favorable decision, the party that files the case and, hence, has a first-mover advantage, or the party that knows more about the set of available judges (Egan, Matvos, and Seru 2021). Indeed, there is widespread advocacy for clear rules to govern case assignments (Transparency International 2007). In practice, however, their successful implementation requires enforcement resources. When enforcement is low, a non-trivial amount of actual assignments could be *irregular*, or inconsistent with the regulations.

From an enforcement policy perspective, this raises the following questions. How many case assignments are irregular? To what extent are irregular assignments made in particular courts or to specific judges? In this paper, I develop tools to address these questions in any given judiciary. Then, I apply them in Ecuador, where multiple irregular assignment scandals have surfaced in 2021, one of which involves the recently deposed mayor of Quito, Ecuador's capital city. I construct a database that contains over 2 million case assignments made in district courts between March 2016 and February 2020, and I detect irregular assignments that are highly localized. 5% of courts and judges account for 43% and 37% of irregular assignments, respectively. Overall, at least 65 thousand assignments, 2.9%, are irregular.

At the heart of these measurements lie the case assignment regulations. They imply that various judicial case characteristics do not influence which judge the case should be assigned to. Examples include the plaintiff's friendly ties with government officials, the amount of money claimed in a payment dispute, or the amount of paperwork that the plaintiff or prosecutor submits when she files the case. If they are sufficiently specific, then they also imply a probability distribution over the judges that each case should be assigned to (e.g. uniform, random assignment). Each implication yields an identification assumption that is informative for the probability that a case's assignment

[1] During that century, Athenian jurors were randomly selected to participate in a given trial using a random, multi-stage selection process that involved an allotment machine called a *kleroterion*. See Dow (1937) p.198.

is irregular, given data on actual assignments. However, the second implication yields a more informative assumption that allows me to measure another parameter of interest: the probability that a case's assignment is irregular, conditional on a given judge. These parameters give a granular view on the structure of irregular assignments and are valuable inputs to guide the allocation of regulatory enforcement resources.

I begin by relating the judge that a case is assigned to with the judge that the case would have been assigned to, had its assignment been irregular, and the judge that the case would have been assigned to, had its assignment been *regulatory*, or in accordance with existing regulations. The counterfactual assignment that is observed will depend on whether the case's assignment is irregular or not. Unlike program evaluation models, where the treatment status is observed, we do not observe any case's irregular assignment status. Indeed, the distribution of the case's irregular assignment status is the object of interest. Thus, our model involves a discrete, two-component mixture.

I then study identification under two assumptions. First, the researcher observes a case characteristic that is statistically independent of counterfactual regulatory assignments only. I call such an instrumental variable a *one-sided instrument*. One-sided instruments are weaker than traditional instrumental variables (e.g. Imbens and Angrist 1994), which require exclusion from both counterfactual outcomes, and their identification power has not been studied in the context of mixture models. They differ from the instrumental variables studied by Henry, Kitamura, and Salanié (2014), which are excluded from observed outcomes, conditional on the unobserved state — the case's irregular assignment status.[2] The second assumption involves a stronger interpretation of the regulations: the probability mass function of counterfactual regulatory assignments is known.

Under each assumption, I obtain analytical solutions for the sharp bounds on the probability that a case's assignment is irregular, as well as the corresponding prob-

[2]They also differ from the mismeasured counterparts used in the literature on data misclassification, which satisfy the same exclusion restriction as the instrumental variables of Henry, Kitamura, and Salanié (2014) (see Bollinger 1996, Mahajan 2006, Y. Hu 2008 and DiTraglia and García-Jimeno 2019).

abilities conditional on covariates (e.g. the case's court and time of assignment) and conditional on the judge that the case is assigned to. This is a two-step process. First, I show that the parameters of interest are examples of linear and scalar parameters whose identified sets are obtained by solving two linear programming problems. Related characterizations include Balke and Pearl (1997), Torgovitsky (2019), Tebaldi, Torgovitsky, and Yang (2019) and Lafférs (2019b). Then, I reformulate the linear programs that define the lower bounds for our parameters of interest in terms of optimal transport problems with convenient graphical interpretations and obtain their closed-form solutions.

The closed-form solutions for the bounds are simple. If the distribution of counterfactual regulatory assignments is known, the lower bound on the probability of an irregular assignment equals the absolute (ℓ_1, taxicab, Manhattan) distance between this distribution and that of actual assignments. With a binary one-sided instrument, this lower bound equals the absolute distance between the two conditional distributions of actual assignments, weighted by the mass of the smaller group, defined in terms of the instrument values. Moreover, knowledge of the distribution of counterfactual regulatory assignments implies that the probability that a case's assignment is irregular, conditional on a given judge, is greater than the rate of cases that the judge received in excess of what she should have received.

I then apply these findings in Ecuador, a country with a GDP per capita that is roughly ten times smaller than that of the U.S. (2019, World Bank), where multiple scandals involving manipulated assignments of cases to judges have surfaced in recent months. According to Ecuadorian regulations, the set of judges that a case can be assigned to depends on the case's location and field of law (i.e. whether the case pertains to criminal law, family law, administrative law, etc...). Within a court, judges are selected on the basis of a lottery system that is not fully specified.

My primary data source is the public, plain text version of the government's database of judicial cases, available on a government website that facilitates individual case searches. I collected and structured the case assignment information for all cases that are contained in this website, to assemble a database with over two million case assignments, performed between March 2016 and February 2020 in Ecuador's 331 district courts.

The one-sided instrument that I consider is the amount of paperwork that the plaintiff or prosecutor submits when she files the case. Its exogeneity stems from the fact that cases with large or small amounts of plaintiff or prosecutor paperwork should not be assigned to judges differently. This instrument alone reveals that at least $9,347$ case assignments, 0.8%, are irregular. In particular, one court accounts for over a third of these assignments.

A more specific interpretation of the regulations is that cases should be assigned to each of the competent judges in any given court with equal probabilities. I assume that a judge is competent for a given case if she is in an active spell when the case is assigned, and if she has a small enough case workload, compared with her peers. A scalar parameter governs each criterion and I select the parameters so as to obtain a conservative lower bound on the probability that a case's assignment is irregular. This exercise implies that 5% of courts and judges account for 43% and 37% of irregular assignments, respectively, and that at least 65 thousand assignments, 2.9%, are irregular. Moreover, 111 judges out of 1568 received cases with irregular assignments.

The primary contribution of this paper is to develop tools to quantify the extent of irregular assignments of cases to judges on the basis of existing regulations and observed assignments. My setting is close to that of Daljord, Pouliot, Xiao, and M. Hu (2021), who measure the extent of black market trade of Beijing license plates under a local government rationing policy. When the distribution of counterfactual regulatory assignments is known, the lower bound on the probability that a case's assignment is irregular equals their optimal transport estimator of the lower bound on the probability that a license plate is traded in the black market. I build on their analysis by introducing one-sided instruments as a means to estimate the same parameter without imposing knowledge of the distribution of a counterfactual outcome. Second, I show that their optimal transport estimator equals the sharp lower bound on the parameter of interest when knowledge of the distribution of one counterfactual outcome is imposed.

My application to Ecuador's judiciary showcases the practical value of these tools to quantify behavior that is typically hard to measure. Indeed, early studies of government corruption (Reinikka and Svensson 2004, Fisman and Wei 2004, Olken 2006) rely on access to the joint distribution of actual outcomes and a potentially noisy measure of the outcomes that would have been observed, had there been no corruption.

From an econometric point of view, this paper introduces one-sided instruments to study non-parametric identification of mixture models (e.g. Hall and Zhou (2003), Henry, Kitamura, and Salanié (2014), Compiani and Kitamura (2016), Kitamura and Laage (2018)). My linear programming formulation of the identified set for the parameters of interest can be seen as an application of Lafférs (2019b), is inspired by Tebaldi, Torgovitsky, and Yang (2019), and is related with Balke and Pearl (1997), Lafférs (2013), Demuynck (2015), Lafférs (2019a) and Torgovitsky (2019). In my setting, I do not observe a proxy variable for the cases' irregular assignment statuses, a common feature in the data misclassification literature (e.g. Bollinger 1996, Mahajan 2006, Molinari 2008, Y. Hu 2008 and DiTraglia and García-Jimeno 2019), I do not have access to the irregular assignment status for a subset of cases, as in Molinari (2010), nor can I credibly set an upper bound on the probability that a case's assignment is irregular, as in Horowitz and Manski (1995).

I organize the paper as follows. Section 1 introduces the econometric framework in a stylized environment and presents the identification results. Section 2 develops the theory of identification that underlies the results presented in Section 1. Section 3 discusses the Ecuadorian context and the available data. Section 4 adapts the econometric framework to the Ecuadorian context and discusses estimation. Section 5 presents the estimation results and section 6 concludes.

1 Illustrative Framework

This section illustrates my identification results in a stylized econometric framework. The framework forms the basis for the empirical model that I use to measure irregular assignments in Ecuador.

1.1 Setting

Consider a stylized setting where a number of judicial cases, indexed by i, are assigned to one of n_Y judges who worked in a given court during a specified time period (e.g. a quarter). Let Y_i denote the judge that case i is assigned to. Label judges from 1 to n_Y, so that Y_i is an observed random variable that takes values in $\{1, \ldots, n_Y\}$.

In this setting, there exist regulations that specify how cases should be assigned to these judges. For example, regulations could mandate simple random assignment, or simple random assignment among a subset of judges. In practice, however, case i's assignment may be *irregular*, or inconsistent with the regulations. Let S_i indicate if i's assignment is irregular or not. This is a latent, binary random variable.

Irregular assignments can arise for various reasons, which I will not attempt to distinguish at this stage. Some reasons, such as administrative errors, do not necessarily involve illegal behavior; others, such as transactions in the black market for judges, do; and some may involve behavior whose legal status is unclear, as with judge shopping, the practice of filing and withdrawing the same case multiple times until the case is assigned to the desired judge.

Consider two counterfactual assignments for any given case. The first counterfactual assignment is the judge that case i would have been assigned to, had its assignment been irregular, $S_i = 1$. The second one is the judge that case i would have been assigned to, had its assignment been *regulatory*, or consistent with the regulations, $S_i = 0$. We denote counterfactual irregular assignments with variable $Y_i(1)$ and counterfactual regulatory assignments with variable $Y_i(0)$. They relate to actual assignments Y_i according to the potential outcomes equation:

$$Y_i \quad = \quad S_i Y_i(1) \ + \ (1 - S_i) Y_i(0). \tag{1}$$

That is, the judge that case i is assigned to equals $Y_i(1)$ if i's assignment is irregular ($S_i = 1$), and equals $Y_i(0)$ otherwise.

For the sake of illustration, I implicitly condition on case i's covariates. I am interested in two parameters: the rate of irregular assignments, and the judge-specific rates of irregular assignments: $\Pr(S_i = 1)$ and $\Pr(S_i = 1 \,|\, Y_i = y^*)$ for each $y^* \in \{1, \ldots, n_Y\}$, respectively. In this setting, these parameters offer a detailed view of the extent and structure of irregular assignments. They are policy-relevant, since they inform the allocation of regulatory enforcement resources.

To measure these quantities, we need assumptions. No component on the right-hand side of (1) is observed. Thus, it is possible that $\Pr\big(Y_i = Y_i(1)\big) = 1$ and $\Pr(S_i = 1) = \Pr(S_i = 1 \,|\, Y_i = y^*) = 1$, for all $y^* \in \{1, \ldots, n_Y\}$. Similarly, it is possible that $\Pr\big(Y_i = Y_i(0)\big) = 1$ and $\Pr(S_i = 1) = \Pr(S_i = 1 \,|\, Y_i = y^*) = 0$, for all $y^* \in \{1, \ldots, n_Y\}$.

I consider two assumptions: that the distribution of regulatory assignments is known, and that the researcher observes a case characteristic Z_i with finite support $\mathcal{Z}$ that does not influence the judge that the case would have been assigned to, had the case's assignment been regulatory.

Assumption PMF. *The probability mass function of* $Y_i(0)$ *is known.*

Assumption IV. Z_i *is statistically independent of* $Y_i(0)$.

I now introduce my identification results for the parameters of interest under each assumption, in turn. At a conceptual level, the discussion of the identification results under Assumption PMF lays the groundwork to introduce the results under Assumption IV.

1.2 Identification Results under Assumption PMF

Assumption PMF states that the distribution of regulatory assignments is known. In the context of my application, I interpret Ecuadorian assignment regulations to mean that, within the court where case i is assigned, i's judge is drawn from a uniform distribution defined over the set of judges that are competent for case i at the time of i's assignment.

Consider the task of measuring the rate of irregular assignments, $\Pr(S_i = 1)$. According to (1), if case i's actual and regulatory assignments differ ($Y_i \neq Y_i(0)$), then i's assignment must be irregular ($S_i = 1$). This means that

$$\Pr(S_i = 1) \;\geq\; \Pr(Y_i \neq Y_i(0)).$$

$\Pr(Y_i \neq Y_i(0))$ is not identified, since the joint distribution of $(Y_i, Y_i(0))$ is unknown. However, we know the marginal distribution of Y_i, since actual assignments are observable, as well as the marginal distribution of $Y_i(0)$, under Assumption PMF. The unobserved joint distribution of $(Y_i, Y_i(0))$ must be consistent with the known marginal distributions. Therefore, a lower bound on the rate of irregular assignments is the minimum probability that case i's actual and regulatory assignments differ that can be obtained from a joint distribution of $(Y_i, Y_i(0))$ that is consistent with the marginal distributions of Y_i and $Y_i(0)$.

Let Γ be the set of probability mass functions defined over $\{1, \ldots, n_Y\} \times \{1, \ldots, n_Y\}$. To summarize our discussion:

$$\begin{aligned}
\Pr(S_i = 1) \quad &\geq \quad \Pr(Y_i \neq Y_i(0)) \\
&\geq \quad \min_{\gamma \in \Gamma} \sum_{y=1}^{n_Y} \sum_{y_0=1}^{n_Y} 1\{y \neq y_0\}\, \gamma(y, y_0) \qquad \text{subject to:} \qquad (2) \\
&\qquad (i) \sum_{y_0=1}^{n_Y} \gamma(y, y_0) \;=\; \Pr(Y_i = y) \quad \text{for all } y \in \mathcal{Y} \\
&\qquad (ii) \sum_{y=1}^{n_Y} \gamma(y, y_0) \;=\; \Pr(Y_i(0) = y_0) \quad \text{for all } y_0 \in \mathcal{Y}.
\end{aligned}$$

Problem (2) is a discrete optimal transport problem (see Galichon (2016)). Because of its binary cost function, which assigns a cost of one if actual and regulatory assignments differ ($y \neq y_0$) and a cost of zero if they do not, problem 2 is particularly tractable. Indeed, its closed-form solution equals half of the absolute (ℓ_1, Taxicab, Manhattan) distance between the marginal distributions of Y_i and $Y_i(0)$ (see Propositions 4.2 and 4.7 of Levin and Peres (2017) for a textbook treatment):

$$\frac{1}{2}\sum_{y=1}^{n_Y}\Big|\Pr(Y_i = y) - \Pr(Y_i(0) = y)\Big|.$$

A similar reasoning produces a lower bound for the judge-specific rates of irregular assignments. Consider judge $y^* \in \{1, \ldots, n_Y\}$ and parameter $\Pr(S_i = 1 \,|\, Y_i = y^*)$. Model (1) implies that:

$$\Pr(S_i = 1, Y_i = y^*) \;\geq\; \Pr(Y_i \neq Y_i(0), Y_i = y^*).$$

That is, if case i's actual and regulatory assignments differ, then i's assignment is irregular, irrespective of the judge that the case was actually assigned to. $\Pr(Y_i \neq Y_i(0), Y_i = y^*)$ is not identified, since we do not observe the joint distribution of $(Y_i, Y_i(0))$. However, the data on actual assignments and Assumption PMF allow us to place a lower bound on this quantity with the minimum probability that $Y_i \neq Y_i(0)$ and $Y_i = y^*$ that can be obtained from a joint distribution of $(Y_i, Y_i(0))$ that is consistent with the marginal distributions of Y_i and $Y_i(0)$:

$$\begin{aligned}
\Pr(S_i = 1, Y_i = y^*) \quad &\geq \quad \Pr(Y_i \neq Y_i(0), Y_i = y^*) \\
&\geq \quad \min_{\gamma\in\Gamma} \quad \sum_{y_0=1}^{n_Y} 1\{y^* \neq y_0\}\,\gamma(y^*, y_0) \qquad \text{subject to:} \qquad (3) \\
&\qquad (i)\ \sum_{y_0=1}^{n_Y}\gamma(y, y_0) \;=\; \Pr(Y_i = y) \quad \text{for all } y \in \mathcal{Y} \\
&\qquad (ii)\ \sum_{y=1}^{n_Y}\gamma(y, y_0) \;=\; \Pr(Y_i(0) = y_0) \quad \text{for all } y_0 \in \mathcal{Y}.
\end{aligned}$$

Once again, (3) is an optimal transport problem. Its cost function assigns a cost of one if actual and regulatory assignments differ *and* the actual assignment is judge y^*, and assigns zero cost otherwise. In this case, the closed-form solution to this problem is simply the amount of cases assigned to judge y^*, beyond the amount of cases that should have been assigned to judge y^*:

$$\max\Big\{0, \quad \Pr(Y_i = y^*) - \Pr(Y_i(0) = y^*\Big\}.$$

Thus, Assumption PMF will place informative lower bounds on the rates of irregular assignments, to the extent that the distributions of actual and regulatory assignments differ. In contrast, Assumption PMF does not place an informative upper bound on these quantities. Because Assumption PMF does not place any restrictions on the distribution of irregular assignments, $Y_i(1)$, it is consistent with the possibility that assignments coincide with irregular assignments: $\Pr(Y_i = Y_i(1)) = 1$. In this case, every case assignment can be irregular: $\Pr(S_i = 1) = \Pr(S_i = 1 \mid Y_i = y^*) = 1$ for all $y^* \in \{1, \ldots, n_Y\}$. Proposition 1 summarizes the discussion.

Proposition 1. *If Assumption PMF holds, then*

1. $$\frac{1}{2}\sum_{y=1}^{n_Y}\Big|\Pr(Y_i = y) \; - \; \Pr(Y_i(0) = y)\Big| \quad \leq \quad \Pr(S_i = 1) \quad \leq \quad 1.$$

2. *For all* $y^* \in \{1, \ldots, n_Y\}$,

$$\max\left\{0, \; \frac{\Pr(Y_i = y^*) - \Pr(Y_i(0) = y^*)}{\Pr(Y_i = y^*)}\right\} \quad \leq \quad \Pr(S_i = 1 \mid Y_i = y^*) \quad \leq \quad 1.$$

These bounds are sharp.

Proposition 1, proven in Appendix C, asserts that the lower bounds that I have introduced are sharp. This means that, for each bound, there exists a joint distribution of the data, $(Y_i(0), Y_i(1), S_i)$, that is consistent with the known distribution of $Y_i(0)$ and with the distribution of actual assignments under equation (1), and generates the given bound. Informally, this means that more information on the parameters of interest cannot be obtained without more data or further assumptions.

Daljord, Pouliot, Xiao, and M. Hu (2021) first proposed the solution to problem (2) as a lower bound on the quantity of black market transactions of license plates in China, following the introduction of a lottery-based rationing system. In their setting, the observed outcome is the price of the car associated with license plate i, and they use the fact that license plates were supposed to be allocated by a lottery to obtain the distribution of car prices in the absence of a black market. Proposition 1 shows that this lower bound is sharp and, hence, promotes the estimand they propose.

1.3 Identification Results under Assumption IV

Under Assumption IV, one observes case characteristics Z_i which, in conjunction with (1), may generate variation in actual assignments Y_i through S_i and/or $Y_i(1)$ exclusively. Because Z_i is excluded only from $Y_i(0)$, I call it a *one-sided instrument*. It differs from the traditional exclusion restriction (e.g. Imbens and Angrist 1994), whereby the instrument generates variation in assignments through S_i only (i.e. statistical independence holds with respect to $(Y_i(0), Y_i(1))$). It also differs from the exclusion restriction proposed by Henry, Kitamura, and Salanié (2014), which requires that Z_i be independent of Y_i, conditional on S_i.[3]

Case characteristics Z_i are discrete. In my application, Z_i is a binary measure of the amount of paperwork submitted by the plaintiff/prosecutor when she files the case. In support of Assumption IV, I argue that Ecuadorian regulations do not contain specific assignment procedures for cases that differ along this dimension, and that this case characteristic is independent of the case characteristics that determine regulatory assignments. Notice that the traditional exclusion restriction, $Z_i \perp\!\!\!\perp (Y_i(0), Y_i(1))$, is unlikely to hold for Z_i. Indeed, Z_i is presumably correlated with irregular assignments, $Y_i(1)$: plaintiffs that file cases with larger amounts of paperwork may value judge attributes differently from others. Plaintiffs with different preferences over judges would select different judges if they were given the chance to do so.

Consider the task of measuring the rate of irregular assignments, $\Pr(S_i = 1)$, under Assumption IV. I proceed as in the discussion of identification under Assumption PMF. Conditional on instrument realization $z \in \mathcal{Z}$, where $\mathcal{Z}$ is the finite support of Z_i, a case's assignment is irregular if its actual assignment differs from its regulatory assignment, by model (1). This observation yields a lower bound on the rate of irregular assignments conditional on $Z_i = z$:

$$\Pr(S_i = 1 \,|\, Z_i = z) \;\geq\; \Pr(Y_i \neq Y_i(0) \,|\, Z_i = z).$$

[3] An equivalent formulation of this statement in terms of potential outcomes is that Z_i is independent of $Y_i(1)$ within the subpopulation with $S_i = 1$ and that Z_i is independent of $Y_i(0)$ within the subpopulation with $S_i = 0$.

A further lower bound on $\Pr(S_i = 1 \mid Z_i = z)$ is given by the minimum probability that case i's actual and regulatory assignments differ, conditional on $Z_i = z$, that can be obtained from a joint distribution of $(Y_i, Y_i(0)) \mid Z_i = z$ that is consistent with the marginal distributions of $Y_i \mid Z_i = z$ and $Y_i(0) \mid Z_i = z$:

$$
\begin{aligned}
\Pr(S_i = 1 \mid Z_i = z) \quad &\geq \quad \Pr(Y_i \neq Y_i(0) \mid Z_i = z) \\
&\geq \quad \min_{\gamma \in \Gamma} \quad \sum_{y=1}^{n_Y} \sum_{y_0=1}^{n_Y} 1\{y \neq y_0\}\, \gamma(y, y_0) \qquad \text{subject to:} \qquad (4) \\
&\qquad (i) \sum_{y_0=1}^{n_Y} \gamma(y, y_0) \;=\; \Pr(Y_i = y \mid Z_i = z) \quad \text{for all } y \in \mathcal{Y} \\
&\qquad (ii) \sum_{y=1}^{n_Y} \gamma(y, y_0) \;=\; \Pr(Y_i(0) = y_0 \mid Z_i = z) \quad \text{for all } y_0 \in \mathcal{Y}, \\
&= \quad \frac{1}{2} \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \mid Z_i = z) \;-\; \Pr(Y_i(0) = y \mid Z_i = z) \Big|
\end{aligned}
$$

where Γ is the set of probability mass functions defined over $\{1, \ldots, n_Y\} \times \{1, \ldots, n_Y\}$, as before, and the last equality follows from the binary cost structure of optimal transport problem (4). It follows that

$$
\begin{aligned}
\Pr(S_i = 1) \quad &= \quad \sum_{z \in \mathcal{Z}} \Pr(Z_i = z) \Pr(S_i = 1 \mid Z_i = z) \\
&\geq \quad \frac{1}{2} \sum_{z \in \mathcal{Z}} \Pr(Z_i = z) \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \mid Z_i = z) \;-\; \Pr(Y_i(0) = y \mid Z_i = z) \Big| \\
&= \quad \frac{1}{2} \sum_{z \in \mathcal{Z}} \Pr(Z_i = z) \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \mid Z_i = z) \;-\; \Pr(Y_i(0) = y) \Big|,
\end{aligned}
$$

where the last equality follows Assumption IV. This lower bound is not identified, however, since the marginal distribution of $Y_i(0)$ is unknown. A further lower bound that is observable is the lowest possible lower bound that is implied by a distribution of $Y_i(0)$. Let Φ be the set of probability mass functions defined over $\{1, \ldots, n_Y\}$. It follows that

$$
\Pr(S_i = 1) \quad \geq \quad \min_{\phi \in \Phi} \frac{1}{2} \sum_{z \in \mathcal{Z}} \Pr(Z_i = z) \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \mid Z_i = z) \;-\; \phi(y) \Big|. \qquad (5)
$$

When Z_i is binary, so that $\mathcal{Z} = \{0, 1\}$, this problem has a closed-form solution. To see this, let $p_{\min} \equiv \min\{\Pr(Z_i = 0), \Pr(Z_i = 1)\}$. For any $\phi \in \Phi$, it follows that

$$
\begin{aligned}
& \frac{1}{2} \sum_{z \in \mathcal{Z}} \Pr(Z_i = z) \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = z) \; - \; \phi(y) \Big| \\
= \; & \frac{1}{2} \Pr(Z_i = 0) \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 0) \; - \; \phi(y) \Big| \\
& \qquad\qquad + \; \frac{1}{2} \Pr(Z_i = 1) \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 1) \; - \; \phi(y) \Big| \\
\geq \; & \frac{p_{\min}}{2} \left(\sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 0) \; - \; \phi(y) \Big| + \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 1) \; - \; \phi(y) \Big| \right) \\
\geq \; & \frac{p_{\min}}{2} \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 0) \; - \; \Pr(Y_i = y \,|\, Z_i = 1) \Big|,
\end{aligned}
$$

where the last inequality follows from the triangle inequality. Moreover, this quantity is achieved by ϕ^*, where

$$
\phi^*(y) = \begin{cases} \Pr(Y_i = y \,|\, Z_i = 0) & \text{if} \quad \Pr(Z_i = 1) \leq \Pr(Z_i = 0) \\ \Pr(Y_i = y \,|\, Z_i = 1) & \text{otherwise.} \end{cases}
$$

The following proposition summarizes.

Proposition 2. *If Assumption IV holds, then*

1. $\displaystyle \min_{\phi \in \Phi} \frac{1}{2} \sum_{z \in \mathcal{Z}} \Pr(Z_i = z) \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = z) \; - \; \phi(y) \Big| \; \leq \; \Pr(S_i = 1) \; \leq \; 1$

2. *For all* $y^* \in \{1, \ldots, n_Y\}$, $\quad 0 \; \leq \; \Pr(S_i = 1 \,|\, Y_i = y) \; \leq \; 1.$

In addition, if Z_i is binary, so that $\mathcal{Z} = \{0, 1\}$, then

$$
\frac{p_{min}}{2} \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 0) \; - \; \Pr(Y_i = y \,|\, Z_i = 1) \Big| \; \leq \; \Pr(S_i = 1) \; \leq \; 1,
$$

where $p_{min} \equiv \min\{\Pr(Z_i = 0), \Pr(Z_i = 1)\}$. Each of these bounds is sharp.

Proposition 2, proven in Appendix C, shows that binary instruments that yield informative lower bounds on $\Pr(S_i = 1)$ will satisfy two conditions. First, a relevance condition: Z_i must induce variation in assignments for the absolute distance between the conditional assignment distributions to be positive. Second, Z_i must be relatively balanced. This is intuitive: if the mass of cases with $Z_i = 0$ is small, the lower bound on $\Pr(S_i = 1)$ arises when regulatory assignments $Y_i(0)$ are distributed according to $Y_i \mid Z_i = 1$, in which case only a small fraction of cases' assignments may be irregular.

Proposition 2 also shows that Assumption IV does not yield informative bounds on $\Pr(S_i = 1 \mid Y_i = y)$. Intuitively, this follows because, if judge y is assigned a substantial amount of cases with $Z_i = 0$ but no cases with $Z_i = 1$, she could either have several cases with irregular assignments if $\Pr(Y_i(0) = y) = \Pr(Y_i = y \mid Z_i = 1)$, or no cases with irregular assignments if $\Pr(Y_i(0) = y) = \Pr(Y_i = y \mid Z_i = 0)$, and Assumption IV cannot distinguish between these possibilities. Thus, my measurements of judge-specific rates of irregular assignments in Ecuador will necessarily involve Assumption PMF.

Furthermore, Proposition 2 shows that Assumption PMF does not place an informative upper bound on the parameters of interest. The intuition for this result is that Assumption IV, like Assumption PMF, does not restrict the distribution of irregular assignments, $Y_i(1)$, so that it is consistent with $\Pr(Y_i = Y_i(1)) = 1$, and $\Pr(S_i = 1) = \Pr(S_i = 1 \mid Y_i = y^*) = 1$ for all $y^* \in \{1, \ldots, n_Y\}$.

Finally, the bounds that Proposition 2 presents are sharp. This means that, for each bound, there exists a joint distribution of the latent and observed data, $(Y_i(0), Y_i(1), S_i, Z_i)$, that satisfies Assumption IV, is consistent with the observed joint distribution of (Y_i, Z_i) under model (1), and generates the given bound. Informally, this means that more information on the parameters of interest cannot be obtained without more data or further assumptions.

2 Identification Analysis

This section develops the theory of identification that underlies propositions 2 and 1, in a setting with explicit covariates.

In addition to the discrete random variables Y_i and Z_i, the researcher observes case characteristics X_i. X_i is a random vector that takes values in finite set $\mathcal{X}$. In the empirical framework of section 4, these characteristics will be the case's court, field of law and time period of assignment. I reformulate Assumptions IV and PMF as:

Assumption IVx. *Z_i is statistically independent of $Y_i(0)$, conditional on X_i.*

Assumption PMFx. *The probability mass function of $Y_i(0) \,|\, X_i = x$ is known, for all $x \in \mathcal{X}$.*

Note that if X_i is degenerate, then Assumptions IVx and PMFx are identical to Assumptions IV and PMF.

The joint distribution of $\big(Y_i(0), Y_i(1), S_i, Z_i\big)$, conditional on covariates X_i, is the cornerstone of the identification analysis, for three reasons. First, the available data, i.e. the probability mass function of (Y_i, Z_i, X_i), constitute restrictions on this distribution, under equation (1). Second, assumptions IVx and PMFx can be reformulated as restrictions on this distribution. Finally, any feature of the joint distribution of the data that we do not observe, any parameter, can be seen as a function of this distribution. Let $\mathcal{F}$ denote the set of probability mass functions of $\big(Y_i(0), Y_i(1), S_i, Z_i\big)$ conditional on X_i. f denotes a typical element of $\mathcal{F}$, and $f(y_0,\, y_1,\, s,\, z \,|\, x)$ denotes a typical value of f.

I proceed in two steps. First, I obtain the restrictions imposed by our data and assumptions on the primitive conditional distribution, f, to define its identified set. Then, I define the identified sets for the parameters of interest and characterize them.

2.1 Identified set for f

The identified set for f is the set of all distributions in $\mathcal{F}$ that are observationally equivalent under model (1), and are consistent with assumptions IVx and PMFx. $f \in \mathcal{F}$ satisfies observational equivalence under model (1) if:

$$\sum_{y_0,y_1,s} 1\big\{sy_1+(1-s)y_0 = y\big\}\, f\big(y_0,\, y_1,\, s,\, z \,|\, x\big) = \Pr(Y_i = y,\, Z_i = z \,|\, X_i = x) \qquad \forall\, y, z, x. \tag{R_{OE}}$$

In other words, f is observationally equivalent whenever its implied distribution of $(Y_i, Z_i)\,|\,X_i$ under model (1) matches that which is observed. Next, any f that is observationally equivalent is consistent with Assumption IVx if:

$$\sum_{y_1,s} f(y_0,\, y_1,\, s,\, z \,|\, x) \;=\; \Pr(Z_i = z \,|\, X_i = x) \sum_{y_1,s,\tilde{z}} f(y_0, y_1, s, \tilde{z} \,|\, x). \qquad \forall\, y_0, z, x. \tag{R_{IV}}$$

That is, f is consistent with Assumption IVx if its implied distribution of $(Y_i(0), Z_i)\,|\,X_i = x$ equals the product of the implied marginal distributions. Notice that the implied distribution of $Z_i\,|\,X_i$ equals the observed distribution by observational equivalence. Finally, $f \in \mathcal{F}$ is consistent with Assumption PMFx if:

$$\sum_{y_1,s,z} f(y_0,\, y_1,\, s,\, z \,|\, x) \;=\; \Pr(Y_i(0) = y_0 \,|\, X_i = x) \qquad \forall\, y_0, x, \tag{R_{PMF}}$$

where $\Pr(Y_i(0) = y_0 \,|\, X_i = x)$ is known, for all $y_0 \in \{1, \dots, n_Y\}$ and $x \in \mathcal{X}$.

Assumptions IVx and PMFx are associated with identified sets $\mathcal{F}^\star_{\text{IV}}$ and $\mathcal{F}^\star_{\text{PMF}}$, respectively, where

$$\begin{aligned} \mathcal{F}^\star_{\text{IV}} &\equiv \big\{f \in \mathcal{F} \,:\, f \text{ satisfies restrictions } (\text{R}_{\text{OE}}) \text{ and } (\text{R}_{\text{IV}})\big\} \quad \text{and} \\ \mathcal{F}^\star_{\text{PMF}} &\equiv \big\{f \in \mathcal{F} \,:\, f \text{ satisfies restrictions } (\text{R}_{\text{OE}}) \text{ and } (\text{R}_{\text{PMF}})\big\}. \end{aligned}$$

The case where both Assumptions IVx and PMFx are imposed need not be treated separately. Under both assumptions, the distribution of $Y_i(0)\,|\,X_i, Z_i$ equals that of $Y_i(0)\,|\,X_i$, which is known. Hence, both assumptions can be cast as Assumption PMFx with covariates $\widetilde{X}_i = (X_i, Z_i)$.

Description	Parameter of Interest	$c(y_0, y_1, s, z, x)$
Rate of Irregular Assignments	$\Pr(S_i = 1)$	$\Pr(X_i = x) \cdot 1\{s = 1\}$
Judge y^*'s Rate of Irregular Assignments	$\Pr(S_i = 1 \mid Y_i = y^*)$	$\frac{\Pr(X_i = x)}{\Pr(Y_i = y^*)} \cdot 1\{s = 1\} \cdot 1\{y_1 = y^*\}$
Rate of Irregular Assignments, given $X_i = x_0$	$\Pr(S_i = 1 \mid X_i = x_0)$	$1\{x = x_0\} \cdot 1\{s = 1\}$
Rate of Irregular Assignments, given $X_i \in \mathcal{X}_0$	$\Pr(S_i = 1 \mid X_i \in \mathcal{X}_0 \subseteq \mathcal{X})$	$\frac{\Pr(X_i = x)}{\Pr(X_i \in \mathcal{X}_0)} \cdot 1\{x \in \mathcal{X}_0\} \cdot 1\{s = 1\}$

Table 1: Coefficients of the Linear Parameters of Interest

2.2 Identified sets for parameters of interest

I cast parameters as linear functions of distributions in $\mathcal{F}$, $\theta : \mathcal{F} \mapsto \mathbb{R}^{d_\theta}$, where d_θ is the dimensionality of parameter θ. Each parameter $\theta = (\theta_1, \ldots, \theta_{d_\theta})$ that I consider is associated with d_θ vectors of known non-negative coefficients $c = (c_1, \ldots, c_{d_\theta})$, so that

$$\theta(f;\, c) \quad \equiv \quad \begin{pmatrix} \sum_{y_0,y_1,s,z,x} c_1(y_0, y_1,\, s,\, z,\, x)\; f(y_0, y_1,\, s,\, z \mid x) \\ \vdots \\ \sum_{y_0,y_1,s,z,x} c_{d_\theta}(y_0, y_1,\, s,\, z,\, x)\; f(y_0, y_1,\, s,\, z \mid x) \end{pmatrix}.$$

When θ is scalar, $\theta(f;\, c) \;\equiv\; \sum_{y_0,y_1,s,z,x} c(y_0, y_1,\, s,\, z,\, x)\;\, f(y_0, y_1,\, s,\, z \mid x)$. The identified set for parameter $\theta(\cdot\,;\, c)$ is the set of parameter values that are associated with distributions that belong to the identified set for f:

$$\begin{aligned} \Theta^\star_{\text{IV}}(c) \quad &\equiv \quad \{\theta(f\,;\, c) : f \in \mathcal{F}^\star_{\text{IV}}\} \quad \text{and} \\ \Theta^\star_{\text{PMF}}(c) \quad &\equiv \quad \{\theta(f\,;\, c) : f \in \mathcal{F}^\star_{\text{PMF}}\}. \end{aligned}$$

Table 1 shows that all of our parameters of interest are linear and presents the associated vectors of coefficients.[4]

[4]In fact, linear parameters are widespread. See, e.g. Mogstad, Santos, and Torgovitsky (2018). For example, the expectation of counterfactual outcome $Y_i(1)$ is the linear parameter associated with coefficients c^1, where $c^1(y_0, y_1, s, z, x) = y_1$; the "Average Treatment Effect" — the average difference between $Y_i(1)$ and

I now turn to the characterization, or computation, of identified sets. Notice first that $\mathcal{F}^{\star}_{\text{IV}}$ and $\mathcal{F}^{\star}_{\text{PMF}}$ are convex sets: the convex combination of any two elements of $\mathcal{F}^{\star}_{\text{IV}}$ (or $\mathcal{F}^{\star}_{\text{PMF}}$) is a well-defined probability mass function that also satisfies restrictions (R_{OE}) and (R_{IV}) (or (R_{PMF})). It is well defined because $\mathcal{F}$, the set of probability mass functions of $\big(Y_i(0), Y_i(1), S_i, Z_i\big)$ conditional on X_i, is convex. It satisfies these restrictions because the solution set to (R_{OE}) and (R_{IV}) (or (R_{PMF})) is convex, which follows from the fact that these restrictions are linear equations in f.

Now, fix non-negative coefficients c and consider parameter $\theta(\cdot\,; c)$. Its identified sets, $\Theta^{\star}_{\text{IV}}(c)$ and $\Theta^{\star}_{\text{PMF}}(c)$, are also convex. In particular, let $f_1, f_2 \in \mathcal{F}^{\star}_{\text{IV}}$. For a given $\lambda \in [0, 1]$, $\lambda f_1 + (1 - \lambda) f_2 \in \mathcal{F}^{\star}_{\text{IV}}$ and

$$\lambda \underbrace{\theta(f_1;\, c)}_{\in \Theta^{\star}_{\text{IV}}(c)} + (1-\lambda) \underbrace{\theta(f_2;\, c)}_{\in \Theta^{\star}_{\text{IV}}(c)} \quad = \quad \theta\big(\lambda f_1 + (1-\lambda) f_2;\, c\big) \;\in\; \Theta^{\star}_{\text{IV}}(c).$$

Thus, the identified set for a scalar and linear parameter under either Assumption IVx or PMFx equals an interval in $\mathbb{R}^{+}$. What is left to determine are the two extreme points of this interval, also known as the *sharp bounds*. But this is straightforward: since the parameter and the restrictions are linear, the extreme points of this interval equal the solution to two linear programming problems that minimize/maximize the parameter value subject to restrictions (R_{OE}), (R_{IV}) and (R_{PMF}). That is, given a vector of non-negative coefficients c, $\Theta^{\star}_{\text{IV}}(c) = \big[\underline{\theta}_{\text{IV}}(c), \overline{\theta}_{\text{IV}}(c)\big]$, where

$$\begin{aligned} \underline{\theta}_{\text{IV}}(c) &= \min_{f \in \mathcal{F}} \quad \theta(f;\, c) \quad \text{subject to } (\text{R}_{\text{OE}}) \text{ and } (\text{R}_{\text{IV}}) \\ \overline{\theta}_{\text{IV}}(c) &= \max_{f \in \mathcal{F}} \quad \theta(f;\, c) \quad \text{subject to } (\text{R}_{\text{OE}}) \text{ and } (\text{R}_{\text{IV}}), \end{aligned}$$

and $\underline{\theta}_{\text{PMF}}(c)$ and $\overline{\theta}_{\text{PMF}}(c)$ are defined analogously.

For our parameters of interest, listed in Table 1, these linear programs either have closed-form solutions or simpler formulations. Table 2 lists the results for parameters $\Pr(S_i = 1 \,|\, X_i = x)$ and $\Pr(S_i = 1 \,|\, Y_i = y^*, X_i = x)$ and Appendix C proves them. Sharp lower bounds for more aggregate parameters such as $\Pr(S_i = 1)$ or $\Pr(S_i =$

$Y_i(0)$ — is the linear parameter associated with c^{ATE}, where $c^{ATE}(y_0, y_1, s, z, x) = y_1 - y_0$; the probability that $Y_i(1)$ (or $Y_i(0)$) equals a given $y \in \mathcal{Y}$ is also a linear parameter. Moreover, the versions of these parameters that condition on $X_i = x$ or $Y_i = y$ are also linear.

Assumption	Parameter of Interest	$\underline{\theta}(c)$	$\overline{\theta}(c)$
IVx	$\Pr(S_i = 1 \mid X_i = x)$	$\min_{\phi \in \Phi} \sum_{z,y} \frac{1}{2} \Pr(Z_i = z \mid x) \Big\vert \Pr(Y_i = y \mid x, z) \; - \; \phi(y \mid x) \Big\vert$, where Φ is the set of p.m.f.s of $Y_i(0) \mid X_i$.	1
IVx	$\Pr(S_i = 1 \mid Y_i = y^*, X_i = x)$	0	1
PMFx	$\Pr(S_i = 1 \mid X_i = x)$	$\frac{1}{2} \sum_{y} \Big\vert \Pr(Y_i = y \mid x) \; - \; \Pr(Y_i(0) = y \mid x) \Big\vert$	1
PMFx	$\Pr(S_i = 1 \mid Y_i = y^*, X_i = x)$	$\max \left\{ 0, \; \frac{\Pr(Y_i = y^* \mid x) - \Pr(Y_i(0) = y^* \mid x)}{\Pr(Y_i = y^* \mid x)} \right\}$	1

Table 2: Sharp Bounds on the Parameters of Interest, conditional on $X_i = x$.

$1 \mid Y_i = y^*)$ can be obtained from the lower bounds listed in Table 2 through appropriate aggregation. In particular, the lower bound for $\Pr(S_i = 1)$ under Assumption PMFx equals:

$$\sum_{x \in \mathcal{X}} \Pr(X_i = x) \left(\frac{1}{2} \sum_{y=1}^{n_Y} \Big\vert \Pr(Y_i = y \mid X_i = x) \; - \; \Pr(Y_i(0) = y \mid X_i = x) \Big\vert \right).$$

3 Context and Data

This section gives an overview of Ecuador's judicial system, discusses the existing regulations on the assignment of cases to judges, and presents the assignment data.

3.1 Context

Unlike federal states, such as Brazil, Mexico, or the United States, Ecuadorian law is homogeneous across its administrative divisions. Ecuador's judiciary has a 3-tiered judiciary, composed of 331 district courts, 24 provincial courts, the National Court of Justice, and a governing body called the Judicial Council. In this paper, I focus on case assignments to judges in the country's district courts.

Table 3 presents the key institutional components that govern the assignment of cases to judges. Lottery offices deployed throughout the country perform assignments. Personnel attached to these offices use a dedicated computer program to draw assignments. In the event of a power outage or any other circumstance where the computer program is not accessible, the personnel draw cases that await assignment sequentially at random and assign them to available judges, who have been arranged in a pre-defined order.[5] Ecuadorian regulations leave the precise implementation of the computer program to the Judicial Council. In a recent interview, however, the president of the Judicial Council briefly explains the implementation: the computer program assigns cases at random among available judges who have a relatively low case workload at the time of assignment. Finally, judges who are available for a given case must work in

[5]This procedure dates from 2004, when assignments were still being performed manually in some Ecuadorian provinces. Since 2013, all case assignments are computer-based by default.

courts that have competence over the case's field of law[6] and location.[7]

Ecuador offers an ideal setting to study irregular assignments of cases to judges, for two reasons. First, this topic is salient and raises concerns among public officials in the Judicial Council, and among the general public. In recent months, several case assignment scandals have surfaced which involve judges in courts across the country as well as high profile individuals, such as the mayor of Quito, the country's capital, who was recently removed from office.[8]

Second, large scale access to case-level assignment information across the country's courts is possible for non-confidential cases, and this information is regularly updated by the Judicial Council.[9] In Latin America, this is exceptional: case-level assignment data is scattered across different judiciaries in federal states such as Mexico or Brazil, and large scale access to case assignment information is effectively denied to the general public in countries such as Argentina, Chile, Colombia, Mexico or Peru.

[6]Each district court has competence over cases that belong to a subset of the following fields of law: criminal law (e.g. a homicide), civil law (e.g. a payment dispute that involves a bank and a credit card debtor), administrative law (e.g. a dispute related with a government contract), tax law (e.g. a tax payment dispute), juvenile law (e.g. a robbery conducted by someone under 18 years of age), transit law (e.g. drunk driving), family violence law (e.g. a case of household violence), family law (e.g. a divorce), labor law (e.g. wrongful termination of an employee) and landlord-tenant law.

[7]The Judicial Council specifies the territory associated with each court. In general, the location of criminal cases is the location where the alleged crime was committed and the location of other cases is the address of the defendant. See article 404 of Código Orgánico Integral Penal 2014, which contains further rules to obtain the jurisdiction if the location of the crime is unknown, and articles 9-15 of Código Orgánico General de Procesos 2015.

[8]See the media coverage here, here, here, and here.

[9]Confidential cases are those that involve sexual crimes, family violence, and crimes against the state. See article 562 of Código Orgánico Integral Penal (2014). Crimes against the state are listed in arts. 336-365. They include rebellion, insubordination of military and police personnel, sabotage, treason, espionage, non-authorized possession of firearms and arms dealing.

Regulation	Original text	Source
The use of the automatic system for case lotteries is compulsory in all districts that have the technological facilities and the system installed.	En los distritos que cuentan con las facilidades tecnológicas y se encuentre instalado el sistema automático de sorteo de causas para primera y segunda instancia, su uso será obligatorio.	Article 9, Reglamento de Sorteo de Juicios (2004)
Districts that do not have the system installed will perform lotteries as follows: after numbering the cases, one ticket for each case is inserted in a container. Tickets are then randomly drawn and determine the judge that the case must be assigned to.	En los distritos o lugares carentes del sistema informático para el sorteo éste tendrá el procedimiento siguiente: Numeradas las demandas o expedientes con arreglo en un recipiente apropiado se colocarán tantas fichas cuantas sean aquellos. Estas fichas se sacarán por la suerte y determinarán a los jueces que deben conocer de las causas.	Article 11, Reglamento de Sorteo de Juicios (2004)
The algorithm of the system assigns cases to judges randomly, according to the judges' case workloads. That is, if we have five judges and each judge has a case workload of ten, then (the system) assigns randomly. But if one of them has a case workload of one hundred, then the system skips that judge, because she has too high a workload	El algoritmo del sistema asigna de manera aleatoria las causas segun la carga procesal que tenga un juzgador. Es decir, si tenemos cinco juzgadores, los cinco tienen carga procesal de diez, entonces va asignando aleatoriamente. Pero si a uno de ellos se le pone una carga procesal de cien causas en trámite, entonces el sistema se salta ese juzgador porque tiene muchas causas en trámite	Minutes 5:48 – 6:30 of an interview with the President of the Judicial Council, available here.

Table 3: Ecuadorian Case Assignment Regulations

Note: English translations are my own.

3.2 Data

My data is a collection of lottery certificates that record individual judicial case assignments to judges. I source this data from http://consultas.funcionjudicial.gob.ec/informacionjudicial/public/informacion.jsf, a website that is maintained by Ecuador's judicial regulator, the *Consejo de la Judicatura*. This public website makes available to the public the government's unique database of judicial cases, called *Sistema Automático de Trámite Judicial Ecuatoriano*.[10]

Every judicial case in the country is given a unique identifier at the time of filing that consists of a two digit number that is associated with each of Ecuador's twenty-four provinces, a three digit number used by the Judicial Council for internal purposes, the four-digit year, and a consecutive number. My data collection exercise requested the information on file for every possible judicial case unique identifier over a period of two months in 2021. For each successful request, I obtained a plain text *.html* file that contained the case's lottery certificate. I then extracted the lottery certificate from this file. For each certificate, I extracted and processed the date when the assignment is produced, the case's reported field of law, the court where the case is assigned, the judge that the case is assigned to, and the amount of paperwork that the plaintiff or prosecutor submits when the case is filed. Figure 1 depicts a lottery certificate, as it appears in the government's webpage.[11]

My data collection exercise returns 2 million lottery certificates that record assignments made in district courts between March, 2016 and February, 2020. I chose the beginning of my sample period for practical reasons: before this time, lottery certificates come in a vast array of formats, which makes the construction of accurate text processing programs a daunting task. I chose to end my sample period before the onset of the SARS-CoV-2 pandemic, which had a sizeable impact on the judiciary's activities, as Figure 2 shows.

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[10] See Machasilla, Mejía, and Torres Feraud (2020) for a description of this database, and articles $118 - 119$ in Código Orgánico General de Procesos (2015) and $578 - 579$ in Código Orgánico Integral Penal (2014) for the legal content requirements of this database.

[11] The black regions conceal the case's identifiable information.

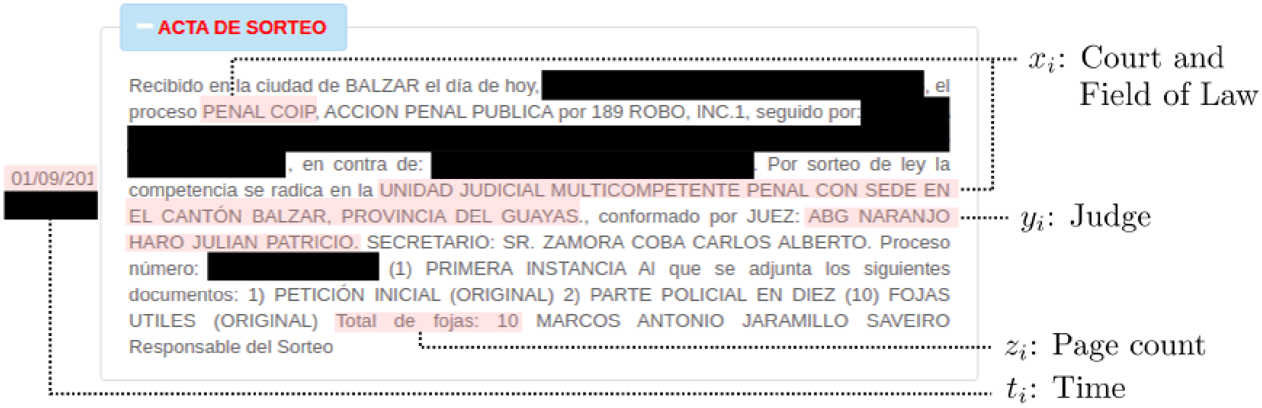


Figure 1: An annotated lottery certificate

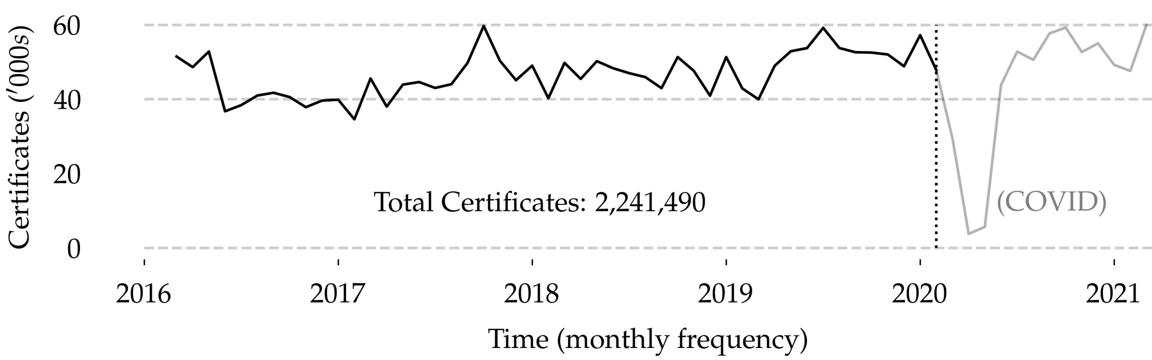


Figure 2: Lottery certificates over time

	Percentiles							
	5	10	25	50	75	90	95	observations
Certificates per Judge	10	72	582	1213	1886	2937	3827	1568
Certificates per Court	17	152	1408	3118	6025	16348	31815	331
Plaintiff Paperwork (number of pages)	0	0	1	5	12	26	44	2, 023, 010

Table 4: Summary Statistics

4 Empirical Framework

This section presents the model that I take to the Ecuadorian setting, adapts the identification results from section 1 and discusses statistical inference.

4.1 Setting

Section 1.1 presented an econometric model of case assignments within a given court, field of law and time period. Now, the model's scope includes every Ecuadorian district court and field of law, between March, 2016 and February, 2020. In section 1.1, data units were judicial cases. Now, they are lottery certificates, although I continue to refer to them as cases.

For each case, the data reveal the case's field of law and the court where it is assigned, X_i, as well as the date and time of assignment, T_i. $n_Y \equiv 1568$ judges worked in Ecuadorian district courts during this time period. Label judges from 1 to n_Y, so that the judge that case i is assigned to, Y_i, is a categorical random variable that takes values in $\{1, \ldots, n_Y\}$. Moreover, I observe the page count of paperwork filed by case i's plaintiff or prosecutor, Z_i. This is a random variable that takes values in finite set $\mathcal{Z}$. In the main specification, Z_i is binary and indicates if the page count exceeds 5, the median page count in the data.

The recent case assignment scandals and my conversations with Judicial Council officials reveal a variety of ways that give rise to irregular assignments. Personnel who work in the case assignment offices may manipulate the Judicial Council's computer program that generates assignments in order to direct assignments. Third parties may infiltrate the Judicial Council's computer system and direct assignments. Judges may call in sick in order to avoid being assigned to specific cases. The cases' locations can be manipulated so as to target certain courts and the cases' fields of law can be misrepresented so as to target certain judges.

I distinguish two classes of irregular assignments: those that involve manipulations of the judge that is selected in the court where the case is assigned and given the case's field of law, and those that involve manipulations of the case's court or field of law. I perform my analysis conditional on X_i, so that I will focus exclusively on the former class of irregular assignments. Therefore, S_i indicates if i's assignment belongs to this class of irregular assignments. This distinction is implicit in the econometric model of section 1, which is a model of case assignments in a given court. Case i's assignment is irregular if $S_i = 1$. Otherwise, it is regulatory. As in section 1, $Y_i(0)$ denotes case i's counterfactual regulatory assignment, and $Y_i(1)$ denotes case i's counterfactual irregular assignment. They relate with actual assignments according to (1).

4.2 Assumptions

Let $\mathcal{T}_i$ be the quarter and year when case i is assigned. Assumption IVe asserts that Z_i is a one-sided instrument, conditional on the case's court, field of law and quarter-year of assignment.

Assumption IVe. *Z_i is independent of $Y_i(0)$, conditional on $(X_i, \mathcal{T}_i)$.*

The motivation for Assumption IVe is that cases with different amounts of plaintiff or prosecutor paperwork should not be assigned to judges differently, irrespective of where and when they are assigned. The threat to Assumption IVe are fluctuations in the composition of plaintiff or prosecutor paperwork within court, field of law and quarter. Judges who are active in times when a large share of incoming cases have extensive plaintiff paperwork would receive cases with more plaintiff paperwork than others. Assumption IVe rules out this compositional variation.

Assumption IVe corresponds with Assumption IVx with covariates X_i and $\mathcal{T}_i$. Thus, rows 1 and 2 of Table 2 list the sharp bounds for $\Pr\left(S_i = 1 \,|\, X_i = x, \mathcal{T}_i = \mathbf{t}\right)$ and $\Pr\left(S_i = 1 \,|\, Y_i = y, X_i = x, \mathcal{T}_i = \mathbf{t}\right)$, respectively, for any x and $\mathbf{t}$.

The second assumption that I consider is that case i's regulatory assignment is uniformly distributed over the set of judges that are competent for i. In a given time and place, a judge is competent for case assignments if she *should* be available for assignments. Appointed judges need not be competent. At times, they may be on a legitimate medical leave or on vacation, for example. Competent judges need not be available: a judge that takes a medical leave so as to avoid a specific case should have been available when the case was assigned.

To state the identification assumption formally, let $\mathcal{J}_{xt}$ be the set of competent judges in court and field of law x at time t. I discuss how I measure this set in appendix section A. Define $\mathcal{T}_i^J$ as the largest time interval that contains case i's time of assignment, T_i, and features the same set of competent judges in X_i, i.e. the joint judge spell when case i was assigned. Concretely, $\mathcal{T}_i^J = \mathcal{T}^J(X_i, T_i)$, where $\mathcal{T}^J(x,t) = \left[\underline{t},\ \bar{t}\right]$ and $\underline{t}, \bar{t}$ satisfy:[12]

i. $t \in \left[\underline{t},\ \bar{t}\right]$

ii. $\bar{t} - \underline{t} \ = \ \sup\left\{\ \bar{\tau} - \underline{\tau}\ :\quad t \in \left[\underline{\tau},\ \bar{\tau}\right] \quad \text{and} \quad \mathcal{J}_{xt} = \mathcal{J}_{x\tau} \text{ for all } \tau \in \left[\underline{\tau},\ \bar{\tau}\right]\right\}.$

$\underline{t}$ and $\bar{t}$ are uniquely defined.[13]

Assumption PMFe. *$Y_i(0)\,|\,X_i = x, \mathcal{T}_i^J = \mathbf{t}\ \sim\ Unif(\mathcal{J}_{xt})$ where $t \in \mathbf{t}$, for all x and $\mathbf{t}$.*

Two features of the institutional setting, listed in Table 3, motivate Assumption PMFe. First, the case assignment procedure that was in place before the implementation of the Judicial Council's computer system, and is still in place when the computer system is out of order implies that cases be assigned to competent judges with equal probabilities. Second, the President of the Judicial Council confirmed in a recent public interview that the computer system draws assignments randomly.

[12] $\underline{t} = \underline{t}(x,t)$ and $\bar{t} = \bar{t}(x,t)$. I omit these arguments for conciseness.

[13] Let $\underline{t}_1, \bar{t}_1$ and $\underline{t}_2, \bar{t}_2$ satisfy i. and ii. Since they satisfy i., $\left[\underline{t}_1, \bar{t}_1\right] \cap \left[\underline{t}_2, \bar{t}_2\right] \neq \varnothing$. Define $\underline{t}^* = \min\{\underline{t}_1, \underline{t}_2\}$ and $\bar{t}^* = \max\{\bar{t}_1, \bar{t}_2\}$. Because $\left[\underline{t}^*, \bar{t}^*\right] = \left[\underline{t}_1, \bar{t}_1\right] \cup \left[\underline{t}_2, \bar{t}_2\right]$,

$$\begin{aligned}\sup\left\{\ \bar{\tau} - \underline{\tau}\ :\quad t \in \left[\underline{\tau},\ \bar{\tau}\right] \quad \text{and} \quad \mathcal{J}_{xt} = \mathcal{J}_{x\tau} \text{ for all } \tau \in \left[\underline{\tau},\ \bar{\tau}\right]\right\} &= \bar{t}^* - \underline{t}^* \\ &\geq \max\left\{\bar{t}_1 - \underline{t}_1,\ \bar{t}_2 - \underline{t}_2\right\}.\end{aligned}$$

Thus, $\underline{t}_1 = \underline{t}_2 = \underline{t}^*$ and $\bar{t}_1 = \bar{t}_2 = \bar{t}^*$.

Assumption PMFe corresponds with Assumption PMFx with covariates X_i and $\mathcal{T}_i^J$. Thus, rows 3 and 4 of Table 2 list the sharp bounds for $\Pr\left(S_i = 1 \,|\, X_i = x, \mathcal{T}_i = \mathbf{t}\right)$ and $\Pr\left(S_i = 1 \,|\, Y_i = y, X_i = x, \mathcal{T}_i = \mathbf{t}\right)$, respectively, for any x and $\mathbf{t}$.

4.3 Statistical Analysis

The bounds presented in sections 1 and 2 are population quantities. In this section, I discuss their measurement in the Ecuadorian context with data on a finite number of judicial cases.

A first task is to decide whether or not irregular assignments of cases to judges occurred within a given set of courts $\mathcal{X}_0$ and time period $\mathcal{T}_0$. That is, we wish to decide between the following hypotheses:

$$H_0 : \Pr(S_i = 1 \,|\, X_i \in \mathcal{X}_0, \mathcal{T}_0) = 0 \qquad \text{vs.} \qquad H_1 : \Pr(S_i = 1 \,|\, X_i \in \mathcal{X}_0, \mathcal{T}_0) > 0. \quad (6)$$

A second task is to measure $\Pr(S_i = 1 \,|\, X_i \in \mathcal{X}_0, \mathcal{T}_0)$.

I use the identification results derived and discussed in sections 1 and 2 to address each of these tasks under Assumptions PMFe and IVe separately, and to address the measurement of judge-specific measures of irregular assignments under Assumption PMFe.

4.3.1 Measuring irregular assignments under Assumption PMFe

Given Assumption PMFe, I obtain bounds on aggregate rates of irregular assignments from the bounds on disaggregate rates shown in Table 2. The rate of irregular assignments within the set of courts $\mathcal{X}_0$ and time periods $\mathcal{T}_0$ satisfies $\Pr(S_i = 1 \,|\, X_i \in \mathcal{X}_0, \mathcal{T}_i \in \mathcal{T}_0) \in [\underline{\theta}_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0), 1]$, where

$$\begin{aligned} \underline{\theta}_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0) &= \sum_{x,\mathbf{t}} \Pr(X_i = x, \mathcal{T}_i^J = \mathbf{t} \,|\, X_i \in \mathcal{X}_0, \mathcal{T}_i^J \in \mathcal{T}_0) \cdot \underline{\theta}_{\text{PMF}}(x, \mathbf{t}) \\ \underline{\theta}_{\text{PMF}}(x, \mathbf{t}) &= \frac{1}{2} \sum_{y \in \mathcal{Y}} \left| \Pr(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t}) - \frac{1\{y \in \mathcal{J}_{x\mathbf{t}}\}}{\#\mathcal{J}_{x\mathbf{t}}} \right|. \end{aligned}$$

Under the hypothesis of no irregular assignments shown in (6), actual and regulatory assignments coincide, so that the distributions of actual and regulatory assignments are identical, so that $\Pr(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t},;\ H_0) = \frac{1\{y \in \mathcal{J}_{x\mathbf{t}}\}}{\#\mathcal{J}_{x\mathbf{t}}}$ for all y, x and $\mathbf{t}$.[14] Hence, the distribution of the test statistic $\widehat{\underline{\theta}}_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0)$, obtained by replacing the unknown probabilities in $\underline{\theta}_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0)$ with their empirical counterparts, can be obtained through repeated simulations of judge assignments made according to the distribution of regulatory assignments. A test of (6) of size α is therefore obtained by rejecting the null hypothesis whenever $\widehat{\underline{\theta}}_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0)$ exceeds the $1-\alpha$ quantile of this distribution.

To measure $\underline{\theta}_{\text{PMF}}(x, \mathbf{t})$, a natural estimator is $\widehat{\underline{\theta}}_{\text{PMF}}(x, \mathbf{t})$. Unfortunately, this estimator need not be unbiased:

$$
\begin{aligned}
&\mathbb{E}\left[\widehat{\underline{\theta}}_{\text{PMF}}(x, \mathbf{t}) - \underline{\theta}_{\text{PMF}}(x, \mathbf{t})\right] \\
= \;& \frac{1}{2}\,\mathbb{E}\Bigg[\sum_{y \in \mathcal{Y}} \left| \widehat{\Pr}(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t}) \;-\; \frac{1\{y \in \mathcal{J}_{x\mathbf{t}}\}}{\#\mathcal{J}_{x\mathbf{t}}} \right| \\
& \qquad\qquad - \left| \Pr(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t}) \;-\; \frac{1\{y \in \mathcal{J}_{x\mathbf{t}}\}}{\#\mathcal{J}_{x\mathbf{t}}} \right| \Bigg] \\
\leq \;& \frac{1}{2}\,\mathbb{E}\Bigg[\sum_{y \in \mathcal{Y}} \left| \widehat{\Pr}(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t}) \;-\; \Pr(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t}) \right| \Bigg] \\
\leq \;& \sqrt{\frac{\#\,\mathrm{supp}(Y_i \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t})}{4\, n_{x\mathbf{t}}}}
\end{aligned}
$$

where $n_{x\mathbf{t}}$ is the number of case assignments that we observe in court x and time period $\mathbf{t}$. The first inequality follows from the triangle inequality for ℓ_1 distances and the second inequality follows from Lemma 5 of Berend and Kontorovich (2013). That $\widehat{\underline{\theta}}_{\text{PMF}}(x, \mathbf{t})$ can be a positively biased estimator of $\underline{\theta}_{\text{PMF}}(x, \mathbf{t})$ is easily seen in the case where there are no irregular assignments, so that $\underline{\theta}_{\text{PMF}}(x, \mathbf{t}) = 0$ but $\Pr\left(\widehat{\underline{\theta}}_{\text{PMF}}(x, \mathbf{t}) > 0\right) > 0$. A bias-corrected estimator of $\underline{\theta}_{\text{PMF}}(x, \mathbf{t})$ is then:

$$
\widehat{\underline{\theta}}^{*}_{\text{PMF}}(x, \mathbf{t}) \quad = \quad \widehat{\underline{\theta}}_{\text{PMF}}(x, \mathbf{t}) \;-\; \sqrt{\frac{\#\,\mathrm{supp}(Y_i \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t})}{4\, n_{x\mathbf{t}}}},
$$

and, for any given set of courts and time period $(\mathcal{X}_0, \mathcal{T}_0)$,I propose to estimate $\underline{\theta}_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0)$

[14] One can verify this by noticing that, $\underline{\theta}_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0) = 0$ under H_0.

with:

$$\widehat{\underline{\theta}}^*_{\text{PMF}}(\mathcal{X}_0, \mathcal{T}_0) \quad \equiv \quad \sum_{x,\mathbf{t}} \widehat{\Pr}(X_i = x, \mathcal{T}_i^J = \mathbf{t} \,|\, X_i \in \mathcal{X}_0, \mathcal{T}_i^J \in \mathcal{T}_0) \;\cdot\; \widehat{\underline{\theta}}^*_{\text{PMF}}(x, \mathbf{t}).$$

Assumption PMFe places informative lower bounds on judge-specific rates of irregular assignments. According to Table 2, the rate of irregular assignments made to judge $y \in \mathcal{Y}$ satisfies $\Pr(S_i = 1 \,|\, Y_i = y) \in [\underline{\theta}_y, 1]$, where

$$\underline{\theta}_y \;\equiv\; \sum_{x,\mathbf{t}} \Pr(X_i = x, \mathcal{T}_i^J = \mathbf{t}) \cdot \max\left\{0, \quad \frac{\Pr(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t}) - \frac{1\{y \in \mathcal{J}_{x\mathbf{t}}\}}{\# \mathcal{J}_{x\mathbf{t}}}}{\Pr(Y_i = y \,|\, X_i = x, \mathcal{T}_i^J = \mathbf{t})}\right\}.$$

To decide if judge y received irregular assignments or not, consider the following hypotheses:

$$H_0: \; \Pr(S_i = 1 \,|\, Y_i = y) = 0 \qquad \text{vs.} \qquad H_1: \; \Pr(S_i = 1 \,|\, Y_i = y) > 0. \tag{7}$$

H_0 in (7) implies that $\theta_y = 0$, in which case $\Pr(Y_i = y) \leq \Pr(Y_i(0) = y)$. Suppose that our data is a realization of n independent and identically distributed random vectors, $\left((Y_i, X_i, \mathcal{T}_i^J)\right)_{i=1}^n$. I consider test statistic $T_y \equiv \widehat{\Pr}(Y_i = y) - \widehat{\Pr}(Y_i(0) = y)$, obtained by replacing the unknown population probabilities in $\Pr(Y_i = y) - \Pr(Y_i(0) = y)$ with their empirical counterparts. The asymptotic distribution of T_y under the assumption that $\Pr(Y_i = y) - \Pr(Y_i(0) = y) = 0$ is given by an application of the classical central limit theorem and the delta method. Thus, I reject H_0 if T_y exceeds the $1 - \alpha$ quantile of this distribution, where α is the desired size of the test.

In practice, we wish to conduct the preceding test of hypotheses for every judge. I implement the procedure from Romano and Wolf (2016) to obtain the $p-$values associated with each of these tests that account for multiple hypothesis testing.

4.3.2 Measuring irregular assignments under Assumption IVe

Given Assumption IVe, the aggregate rates of irregular assignments can be derived from Table 2. That is, the rate of irregular assignments within the set of courts $\mathcal{X}_0$ and quarters $\mathcal{T}_0$ satisfies $\Pr(S_i = 1 \mid X_i \in \mathcal{X}_0, \mathcal{T}_i \in \mathcal{T}_0) \in [\underline{\theta}_{\text{IV}}(\mathcal{X}_0, \mathcal{T}_0), 1]$, where[15]

$$\begin{aligned}
\underline{\theta}_{\text{IV}}(\mathcal{X}_0, \mathcal{T}_0) &= \sum_{x,\mathbf{t}} \Pr(X_i = x, \mathcal{T}_i = \mathbf{t} \mid X_i \in \mathcal{X}_0, \mathcal{T}_i \in \mathcal{T}_0) \cdot \underline{\theta}_{\text{IV}}(x, \mathbf{t}) \\
\underline{\theta}_{\text{IV}}(x, \mathbf{t}) &= \min_{\phi \in \Phi} \sum_{z \in \mathcal{Z}, y \in \mathcal{Y}} \Bigg(\frac{\Pr(Z_i = z \mid X_i = x, \mathcal{T}_i = \mathbf{t})}{2} \\
&\qquad \times \Big| \Pr(Y_i = y \mid Z_i = z, X_i = x, \mathcal{T}_i = \mathbf{t}) - \phi(y \mid x, \mathbf{t}) \Big| \Bigg)
\end{aligned}$$

and Φ is the set of probability mass functions defined over $\mathcal{Y}$.

Under the hypothesis of no irregular assignments shown in (6), $\underline{\theta}_{\text{IV}}(\mathcal{X}_0, \mathcal{T}_0) = 0$. As long as the one-sided instrument has full support within each court and quarter contained in $(\mathcal{X}_0, \mathcal{T}_0)$, $\underline{\theta}_{\text{IV}}(\mathcal{X}_0, \mathcal{T}_0) = 0$ implies that actual assignments Y_i are statistically independent of the one-sided instrument Z_i, conditional on $\{X_i = x, \mathcal{T}_i = \mathbf{t}\}$, for all $x \in \mathcal{X}_0$ and $\mathbf{t} \in \mathcal{T}_0$. I propose to use this implication to justify the randomization hypothesis that underpins a permutation test of (6).

To introduce the test, suppose that we observe the assignment information of n judicial cases, so that the available data is a realization of the discrete random vector $W \equiv \big((Y_i, Z_i, X_i, \mathcal{T}_i)\big)_{i=1}^n$ that takes values in set $\mathcal{W}$. Given the set of courts $\mathcal{X}_0$ and quarters $\mathcal{T}_0$, consider a map $g : \mathcal{W} \mapsto \mathcal{W}$ such that:

$$\begin{aligned}
& g\big((y_1, z_1, x_1, \mathbf{t}_1), (y_2, z_2, x_2, \mathbf{t}_2), \dots (y_n, z_n, x_n, \mathbf{t}_n)\big) \\
= \; & \big((y_1, z_{\gamma(1)}, x_1, \mathbf{t}_1), (y_2, z_{\gamma(2)}, x_2, \mathbf{t}_2), \dots (y_n, z_{\gamma(n)}, x_n, \mathbf{t}_n)\big),
\end{aligned}$$

where

$$\gamma(i) = \begin{cases} \pi(i) & \text{if } x_i \in \mathcal{X}_0 \text{ and } \mathbf{t}_i \in \mathcal{T}_0 \\ i & \text{otherwise} \end{cases}$$

and π is a permutation on $\{i : \; x_i \in \mathcal{X}_0 \text{ and } \mathbf{t}_i \in \mathcal{T}_0\}$. Let G be the set of all such maps.[16]

Proposition 3. *Fix a set of courts $\mathcal{X}_0$ and quarters $\mathcal{T}_0$ and consider hypothesis H_0 : $\Pr(S_i = 1 \mid X_i \in \mathcal{X}_0, \mathcal{T}_i \in \mathcal{T}_0) = 0$. If (i) Assumption IVe holds, (ii) $\big((Y_i, Z_i, X_i, \mathcal{T}_i)\big)_{i=1}^n$*

[15]Recall that Z_i is a discrete random variable that takes values in (finite) set $\mathcal{Z}$.

[16]I omit the dependence of G on $(\mathcal{X}_0, \mathcal{T}_0)$ for notational brevity.

is independent and identically distributed under H_0*, and (iii)* $\Pr(Z_i = z \,|\, X_i = x, \mathcal{T}_i = \mathbf{t}) > 0$ *for all* $z \in \mathcal{Z}$ *and for every* $x \in \mathcal{X}_0$ *and* $\mathbf{t} \in \mathcal{T}_0$ *under* H_0*, then* W *and* $g(W)$ *are identically distributed for all* $g \in \mathcal{G}$ *under* H_0*.*

Proof. Fix $w = \big((y_1, z_1, x_1, \mathbf{t}_1), \ldots, (y_n, z_n, x_n, \mathbf{t}_n)\big) \in \mathcal{W}$ and $g \in \mathcal{G}$. Let $\mathcal{I}_w \equiv \{i \in \{1, \ldots, n\} : \; x_i \in \mathcal{X}_0, \mathbf{t}_i \in \mathcal{T}_0\}$ and denote by π the permutation of $\mathcal{I}_w$ associated with g. Under H_0, it follows that

$$
\begin{aligned}
& \Pr(W = w) \\
= \; & \Pr\big((Y_i, Z_i, X_i, \mathcal{T}_i) = (y_i, z_i, x_i, \mathbf{t}_i) \; \forall i \notin \mathcal{I}_w\big) \\
& \times \prod_{i \in \mathcal{I}_w} \Pr\big((X_i, \mathcal{T}_i) = (x_i, \mathbf{t}_i)\big) \Pr\big(Y_i = y_i \,|\, x_i, \mathbf{t}_i\big) \Pr\big(Z_i = z_i \,|\, x_i, \mathbf{t}_i\big) \\
= \; & \Pr\big((Y_i, Z_i, X_i, \mathcal{T}_i) = (y_i, z_i, x_i, \mathbf{t}_i) \; \forall i \notin \mathcal{I}_w\big) \\
& \times \prod_{i \in \mathcal{I}_w} \Pr\big((X_i, \mathcal{T}_i) = (x_i, \mathbf{t}_i)\big) \Pr\big(Y_i = y_i \,|\, x_i, \mathbf{t}_i\big) \Pr\big(Z_i = z_{\pi^{-1}(i)} \,|\, x_i, \mathbf{t}_i\big) \\
= \; & \Pr\big(W = g^{-1}(w)\big) \\
= \; & \Pr\big(g(W) = w\big),
\end{aligned}
$$

where the first equality uses the fact that H_0 and our full support assumption on Z_i imply that $Y_i \perp\!\!\!\perp Z_i \,|\, X_i = x, \mathcal{T}_i = \mathbf{t}$ for all $x \in \mathcal{X}_0$ and $\mathbf{t} \in \mathcal{T}_0$. ■

Proposition 3 states the conditions under which the randomization hypothesis holds, i.e. the distribution of our data is invariant to transformations in $\mathcal{G}$ under the null hypothesis. To test (6), consider the test statistic $\widehat{\underline{\theta}}_{\text{IV}}(\mathcal{X}_0, \mathcal{T}_0)$, obtained by replacing the unknown probabilities in $\underline{\theta}_{\text{IV}}(\mathcal{X}_0, \mathcal{T}_0)$ with their empirical counterparts. Under the randomization hypothesis, the quantiles of the set of test statistic values obtained across possible transformations of the data $g \in \mathcal{G}$ serve as the quantiles of the distribution of the test statistic under the null hypothesis to test (6) at a given size α. For a textbook treatment of permutation tests, see section 15.2 in Lehmann and Romano (2005).

5 Results

This section describes my estimates of irregular assignments in Ecuador. The one-sided instrument detects 9,347 irregular assignments, 0.46% of total assignments. Four courts account for over 50% of these irregular assignments. When I impose knowledge of the distribution of regulatory assignments (Assumption PMFe), I detect more irregular assignments, in more courts and among 7% of judges.

Table 5 shows the estimated lower bounds on the overall overall rate of irregular assignments and the rates of irregular assignments for civil and criminal cases separately. Across specifications, column (1) shows lower bounds of around six percent. These lower bounds have substantive positive biases, however. The magnitude of the bias can be seen in columns (2), (3) and (4), which give the fifth, median and ninety-fifth quantiles of the distributions of the estimators under the null hypothesis of an uninformative lower bound. After subtracting the median of the distribution of the estimators under the null hypothesis from the point estimates, I find that the one-sided instrument detects over 9 thousand irregular assignments, exclusively among criminal cases. On the other hand, knowledge of the distribution of regulatory assignments detects irregular assignments for both criminal and civil cases, and implies at least 64,929 irregular assignments overall.

Figure 3a shows that the one-sided instrument's detections single out a handful of courts. One court accounts for a third of all irregular assignments detected. In that court, at least 16% of case assignments are irregular. Moreover, 5 courts account for over 50% of all assignments detected. Since cases with large amounts of prosecutor paperwork are sometimes assigned to judges differently from cases with small amount of prosecutor documentation, this finding suggests that prosecutors of criminal cases in these courts are involved in irregular assignments. Figure 3b shows that Assumption PMFe also detects irregular assignments in these courts, which are depicted as the colored dots.

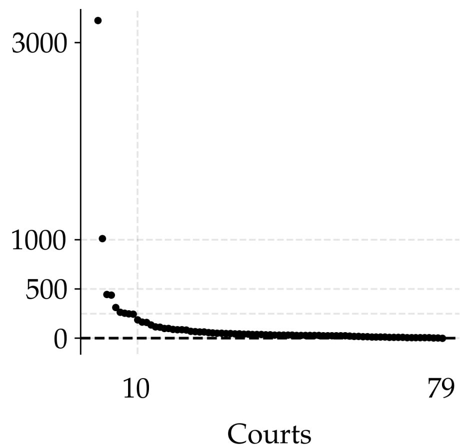
Finally, Figure 3c shows the lower bounds on the judge-specific rates of irregular assignments. I reject the null hypothesis of an uninformative lower bound for 111 judges, out of 1538, using $p-$values that are corrected for multiple hypothesis testing (Romano and Wolf 2016). Four judges stand out from the rest. Anecdotally, the judge with the highest lower bound depicted in Figure 3c faced corruption charges in March of 2020, shortly after the end of my sample period.

	$\widehat{LB}$	$\widehat{LB}$ percentiles under $H_0 : LB = 0$					
	$\widehat{LB}$	$p5$	$p50$	$p95$	$\widehat{LB} - p50$	$\hat{n}$ Irregular	Observations
	(1)	(2)	(3)	(4)	(5)	(6)	(7)
Panel A: Estimates Based on Assumption IVe							
Overall	6.05***	5.57	5.59	5.62	0.46	9,347	2,023,010
By Field of Law:							
Civil	6.19	6.14	6.18	6.22	0.00	39	937,765
Criminal	5.94***	5.05	5.08	5.13	0.86	9,303	1,085,245
Panel B: Estimates Based on Assumption PMFe							
Overall	6.13***	3.20	3.23	3.26	2.90	64,929	2,241,490
By Field of Law:							
Civil	6.00***	3.51	3.56	3.60	2.44	26,776	1,096,693
Criminal	6.25***	2.87	2.92	2.96	3.33	38,144	1,144,797

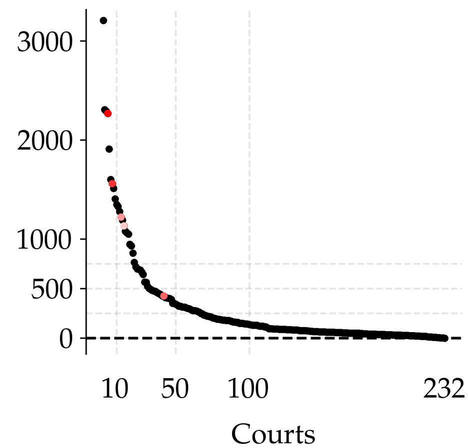
Table 5: Aggregate Lower Bounds on Irregular Assignments

The three rows in Panel A show the estimation results for the lower bounds on $\Pr(S_i = 1)$, $\Pr(S_i = 1 \mid L_i = \text{civil})$ and $\Pr(S_i = 1 \mid L_i = \text{criminal})$ in percentage points, respectively, under Assumption IVe. The three rows in Panel B show the results under Assumption PMFe. Column (1) shows the estimated lower bounds. Columns (2) – (4) show percentiles 5, 50 and 95 of the distribution of the estimator under the null hypothesis of an uninformative lower bound. In Panel A, this distribution is obtained from repeated permutations of the instrument realizations. Column (5) substracts the median of these distributions from the estimated lower bound. Column (6) scales column (5) with the total number of assignments, given in column (7). Significance levels: *** $p < 0.01$.

(a) Irregular Assignments by Court with a One-Sided Instrument



(b) Irregular Assignments by Court with Knowledge of Regular Assignments



(c) Irregular Assignments by Judge with Knowledge of Regular Assignments

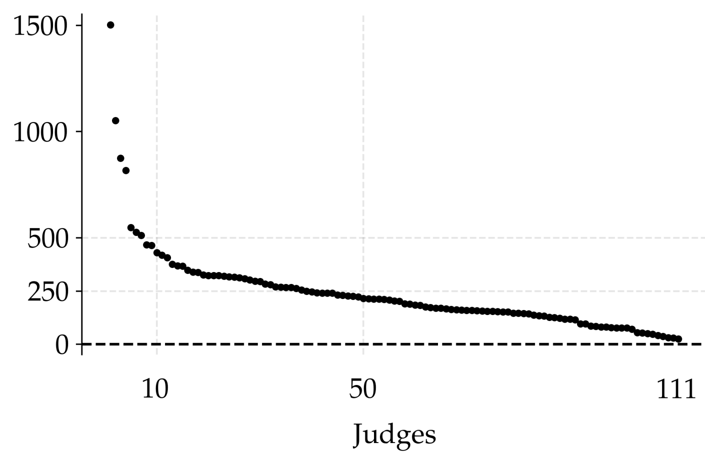


Figure 3: Disaggregate Lower Bounds on Irregular Assignments

The two top figures show the amount of irregular assignments by court, among the courts whose rate of irregular assignments is significant at the family-wise error rate of 5%, under Assumption IVe (figure 3a) and under Assumption PMFe (figure 3b). Irregular assignment amounts are computed as in Table 5: they equal the estimated lower bounds net of the median of the distribution of the estimator under the null hypothesis of an uninformative lower bound, multiplied by the number of assignments made in the given court or to the given judge.

6 Conclusion

In this paper, I developed a method to evaluate a basic aspect of judicial activity: the assignment of cases to judges. The method yielded measurements on the extent to which actual assignments violate the regulations that govern them. In particular, it provided the most informative bounds on the extent of violations that can be achieved with individual case assignment data and knowledge of the existing regulations. Such data is available to the public in Ecuador, but is routinely collected by judiciaries around the world.

In Ecuador, a weak interpretation of the regulations suggested an instrumental variable that detected irregular assignments in a handful of courts. A stronger interpretation of the regulations implied that 7% of judges who worked in district courts between March, 2016 and February, 2020 were involved in such assignments, and that 2.9% of case assignments violated the regulations. In either case, the irregular assignments that I detected are highly localized. These findings suggest that the method is a useful tool to direct regulatory enforcement resources.

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A Measurement of the Set of Competent Judges

Assumption PMFe raises a challenge: which judges are competent in a given court and field of law x, and point in time t? I consider a judge to be competent in (x, t) whenever she is active and has a low relative workload.

Competent judges must be in an active spell. Judge y is in an active spell in court and field of law x at time t if t falls within a window of less than α days between the time when she was last assigned a case in x, and the time when she will be next assigned a case in x. Thus, lower values of α require judges to receive cases at a higher frequency to be considered active. Any choice of α entails two errors. A judge could be considered active at (x, t) when she actually was not; and a judge could be considered inactive at (x, t) when she actually was active. High values of α produce the former errors, whereas low values of α produce the latter errors.

Competent judges must have a case workload that is less than β times the workload of their peer with the lowest case workload at (x, t). I include this criterion because it is listed as such by the President of the Judicial Council (see Table 3).

I select parameters α and β so as to obtain conservative lower bounds on the overall rate of irregular assignments, $\Pr(S_i = 1)$. Figure A.1 shows that I achieve a conservative lower bound on this parameter when I ignore judges' case workloads ($\beta = \infty$), and when judges must receive cases at a frequency of at least $\alpha = 30$ days to be considered active between assignments.

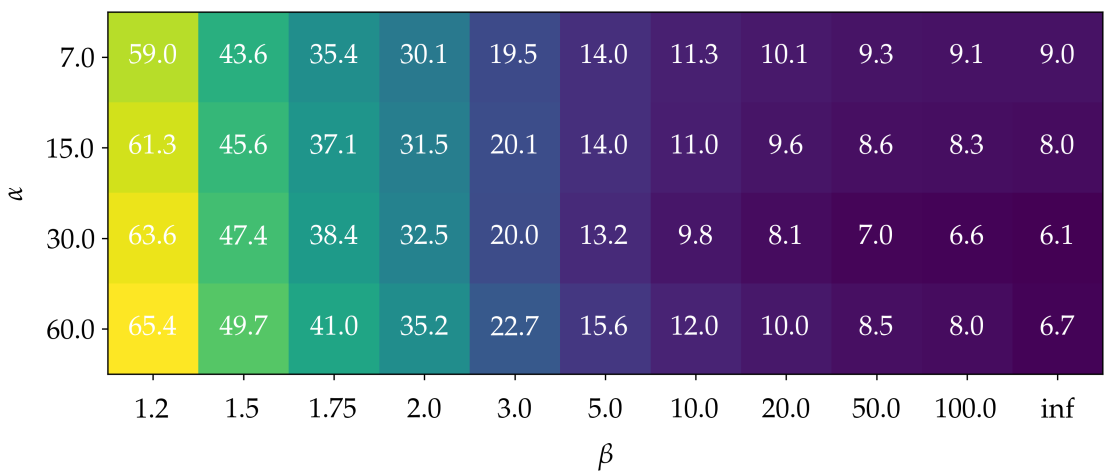

Figure A.1: Lower Bounds on Overall Irregular Assignments across Tuning Parameters

Each cell shows the estimated lower bound on the rate of irregular assignments, $\Pr(S_i = 1)$, under Assumption PMFe for a given value of α and β. α is measured in days and equals the minimum frequency of cases assignments made to a judge for her to be in an active spell. β is the maximum case workload of a judge (relative to the peer with the lowest workload) for her to be available for case assignments. When $\beta = \infty$, judges' workloads do not determine if they are available for case assignments or not.

B Missing Data

In this section, I consider interpretations of the findings in Table 2 that are conscious of missing judicial case assignment information.

In section 2 of the main text, I assume that the unconditional distribution case assignment information, (Y_i, Z_i, X_i), is known or estimable. In the application, however, the data includes the assignment information of publicly-available judicial cases that I retrieved from a website of the Ecuadorian government. Three kinds of judicial cases are therefore missing in my data: non-confidential cases that are unavailable in the government's website, non-confidential cases that are available in the government's website, but were not retrieved by my data collection exercise, and confidential cases.

Let M_i indicate if case i belongs to one of these three mutually-exclusive categories. Given the data-collection procedure, only the distribution of case assignment information among non-missing judicial cases, $(Y_i, Z_i, X_i) \,|\, M_i = 0$, can be known or estimable. The identification results listed in Table 2 are easily adapted to this setting however, provided the following assumptions are made.

Assumption PMFm. *M_i is statistically independent of $Y_i(0)$, conditional on X_i.*

Assumption IVm. *Z_i is statistically independent of $Y_i(0)$, conditional on X_i and $M_i = 0$.*

Assumptions PMFx and PMFm amount to taking the distribution of regulatory assignments, $Y_i(0)$, to be known (conditional on X_i). Assumption PMFm requires that M_i satisfy the one-sided instrument exclusion restriction. In support of this assumption, I note that Ecuadorian regulations do not specify distinct case assignment procedures for confidential cases. Under Assumptions PMFx and PMFm, the identification results in Table 2 hold, conditional on $M_i = 0$: for all $x \in \mathcal{X}$ and $y^* \in \mathcal{Y}$,

$$\begin{aligned} &\Pr(S_i = 1 \,|\, X_i = x, M_i = 0) \qquad\qquad\qquad\qquad \text{(A.1)} \\ &\qquad \in \left[\frac{1}{2} \sum_y \Big| \Pr(Y_i = y \,|\, x, M_i = 0) \;-\; \Pr(Y_i(0) = y \,|\, x) \Big|, \quad 1 \right] \end{aligned}$$

and

$$\Pr(S_i = 1 \,|\, Y_i = y^*, X_i = x, M_i = 0) \tag{A.2}$$
$$\in \left[\max \left\{ 0,\ \frac{\Pr(Y_i = y^* \,|\, x, M_i = 0) - \Pr(Y_i(0) = y^* \,|\, x)}{\Pr(Y_i = y^* \,|\, x, M_i = 0)} \right\},\quad 1 \right].$$

On the other hand, Assumption IVm posits that Z_i is a one-sided instrument, conditional on the judicial case not being missing. Under Assumption IVm, the identification results under one-sided instruments in Table 2 hold conditional on $M_i = 0$: for every $x \in \mathcal{X}$,

$$\Pr(S_i = 1 \,|\, X_i = x, M_i = 0) \tag{A.3}$$
$$\in \left[\min_{\phi \in \Phi} \sum_{z,y} \frac{1}{2} \Pr(Z_i = z \,|\, x, M_i = 0) \Big| \Pr(Y_i = y \,|\, x, z, M_i = 0) \;-\; \phi(y \,|\, x) \Big|,\quad 1 \right].$$

I note that the bounds presented in (A.1), (A.2) and (A.3) are valid for the rates of irregular assignments among all non-confidential cases, not just among non-missing cases, under two additional assumptions: that my data-collection exercise fails to retrieve the information of available judicial cases at random, and that the assignment of a non-confidential judicial case that is not available in the government's website is more likely to be irregular than that the assignment of a non-confidential case that is available in the government's website.

C Proofs

Lemmas 1 – 5 construct the sharp bounds presented in Propositions 1 and 2 on the basis of the theory of identification from Section 2. Notice that Propositions 1 and 2 do not involve case covariates, X_i, for the sake of illustration. This means that the support of X_i, $\mathcal{X}$, is implicitly assumed to be a singleton.

Proof of Proposition 1. By Lemma 1, the sharp upper bound for $\Pr(S_i = 1)$ and the sharp upper bounds for $\Pr(S_i = 1 \mid Y_i = y^*)$ for every $y^* \in \{1, \ldots, n_Y\}$ all equal one. By Lemma 5, the sharp lower bound for $\Pr(S_i = 1)$ is:

$$\sum_{y \in \mathcal{Y}} \frac{1}{2} \Big| \Pr(Y_i = y) \; - \; \Pr(Y_i(0) = y) \Big|.$$

whereas, for any $y^* \in \{1, \ldots, n_Y\}$, the sharp lower bound for $\Pr(S_i = 1 \mid Y_i = y^*)$ is:

$$\max\left\{0, \quad \frac{\Pr(Y_i = y^*) \; - \; \Pr(Y_i(0) = y^*)}{\Pr(Y_i = y^*)}\right\}.$$

■

Proof of Proposition 2. By Lemma 1, the sharp upper bound for $\Pr(S_i = 1)$ and the sharp upper bounds for $\Pr(S_i = 1 \mid Y_i = y^*)$ for every $y^* \in \{1, \ldots, n_Y\}$ all equal one. By Lemma 3, the sharp lower bound for $\Pr(S_i = 1)$ is:

$$\sum_{y \in \mathcal{Y}, z \in \mathcal{Z}} \frac{p_{\min}}{2} \Big| \Pr(Y_i = y \mid Z_i = 0) \; - \; \Pr(Y_i = y \mid Z_i = 1) \Big|$$

whereas, for any $y^* \in \{1, \ldots, n_Y\}$, the sharp lower bound for $\Pr(S_i = 1 \mid Y_i = y^*)$ equals zero. ■

Lemma 1. Fix $x^* \in \mathcal{X}$ and $y^* \in \{1, \dots, n_Y\}$. Consider the parameter associated with $\Pr(S_i = 1 \mid X_i = x^*)$, $\theta(f; c)$, where $c(y_0, y_1, s, z, x) \equiv 1\{x = x^*\} \cdot 1\{s = 1\}$, and the parameter associated with $\Pr(S_i = 1 \mid Y_i = y^*, X_i = x^*)$, $\theta(f; c_y)$, where $c_y(y_0, y_1, s, z, x) \equiv \frac{1\{x=x^*, y_1=y^*\}}{\Pr(Y_i=y^* \mid X_i=x^*)} \cdot 1\{s = 1\}$. Then,

$$\overline{\theta}_{\text{PMF}}(c) = \overline{\theta}_{\text{PMF}}(c_y) = \overline{\theta}_{\text{IV}}(c_y) = \overline{\theta}_{\text{IV}}(c) = 1.$$

Proof. Let Assumption PMFx hold and define the data-generating process f_{PMF} as:

$$f_{\text{PMF}}(y_0, y_1, s, z \mid x) = \begin{cases} 0 & \text{if } s = 0 \\ \Pr(Y_i(0) = y_0 \mid X_i = x) \Pr(Y_i = y_1, Z_i = z \mid X_i = x) & \text{if } s = 1. \end{cases}$$

f_{PMF} is well-defined, since it is weakly positive and adds up to one for each $x \in \mathcal{X}$. Now, $f_{\text{PMF}} \in \mathcal{F}^*_{\text{PMF}}$, since it satisfies restrictions (R_{OE}) and (R_{PMF}). Moreover,

$$\begin{aligned}
\theta\left(f_{\text{PMF}};\ c\right) &= \sum_{y_0, y_1, s, z, x} c(y_0, y_1, s, z, x) f_{\text{PMF}}(y_0, y_1, s, z \mid x) \\
&= \sum_{y_0, y_1, z} f_{\text{PMF}}(y_0, y_1, 1, z \mid x^*) \\
&= \sum_{y_0, y_1, z} \Pr(Y_i(0) = y_0 \mid X_i = x^*) \Pr(Y_i = y_1, Z_i = z \mid X_i = x^*) \\
&= \sum_{y_1, z} \Pr(Y_i = y_1, Z_i = z \mid X_i = x^*) \\
&= 1.
\end{aligned}$$

and

$$\begin{aligned}
\theta\left(f_{\text{PMF}};\ c_y\right) &= \sum_{y_0, y_1, s, z, x} c_y(y_0, y_1, s, z, x) f_{\text{PMF}}(y_0, y_1, s, z \mid x) \\
&= \sum_{y_0, z} \frac{f_{\text{PMF}}(y_0, y^*, 1, z \mid x^*)}{\Pr(Y_i = y^* \mid X_i = x^*)} \\
&= \sum_{y_0, z} \Pr(Y_i(0) = y_0 \mid X_i = x^*) \frac{\Pr(Y_i = y^*, Z_i = z \mid X_i = x^*)}{\Pr(Y_i = y^* \mid X_i = x^*)} \\
&= 1.
\end{aligned}$$

Since $\theta(f; c) \le 1$ and $\theta(f; c_y) \le 1$ for all $f \in \mathcal{F}$, it follows that $\overline{\theta}_{\text{PMF}}(c) = \overline{\theta}_{\text{PMF}}(c_y) = 1$.

Now let Assumption IVx hold and, for a given probability mass function ϕ defined over $\{1,\ldots,n_Y\}$ for each $x \in \mathcal{X}$, define the data-generating process f_{IV} as:

$$f_{\text{IV}}(y_0, y_1, s, z \,|\, x) = \begin{cases} 0 & \text{if } s = 0 \\ \phi(y_0 \,|\, x)\Pr(Y_i = y_1, Z_i = z \,|\, X_i = x) & \text{if } s = 1. \end{cases}$$

Since f_{IV} is weakly positive, adds up to one for every $x \in \mathcal{X}$ and satisfies restrictions (R_{OE}) and (R_{IV}), $f_{\text{IV}} \in \mathcal{F}^*_{\text{IV}}$. Moreover, $\theta\left(f_{\text{IV}}; c\right) = \theta\left(f_{\text{IV}}; c_y\right) = 1$, so that $\overline{\theta}_{\text{IV}}(c) = \overline{\theta}_{\text{IV}}(c_y) = 1$. ■

Lemma 2. Define $\mathcal{Y} \equiv \{1,\ldots,n_Y\}$ and consider the linear parameter $\theta(f; c^S)$ associated with scalar coefficients c^S, such that $c^S(y_0, y_1, s, z, x) \equiv \omega(y_1, x)1\{s = 1\}$ and $\omega : \mathcal{Y} \times \mathcal{X} \mapsto \mathbb{R}$. Let Γ be the set of probability mass functions defined over $\mathcal{Y} \times \mathcal{Y}$ and Φ be the set of probability mass functions defined over $\mathcal{Y}$ for each $x \in \mathcal{X}$. Then

$$\begin{aligned} \underline{\theta}_{\text{IV}}(c^S) \quad &\equiv \quad \min_{f \in \mathcal{F}^*_{\text{IV}}} \quad \theta(f;\, c^S) \\ &= \quad \min_{\phi \in \Phi} \sum_{z \in \mathcal{Z}, x \in \mathcal{X}} \Pr(Z_i = z \,|\, X_i = x)\; \lambda(\phi, z, x) \end{aligned}$$

where

$$\begin{aligned} \lambda(\phi, z, x) \quad = \quad \min_{\gamma \in \Gamma} \sum_{y_0, y \in \mathcal{Y}} & \omega(y, x)\; 1\{y_0 \neq y\}\; \gamma(y_0, y) \qquad \text{s.t.} \\ (i) \quad \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y) \;&=\; \Pr(Y_i = y \,|\, Z_i = z, X_i = x) \qquad \forall\, y \\ (ii) \quad \sum_{y \in \mathcal{Y}} \gamma(y_0, y) \;&=\; \phi(y_0 \,|\, x) \qquad \forall\, y_0 \end{aligned}$$

Proof. The proof has two parts. In Part I, I show that $\underline{\theta}_{\text{IV}}(c^S)$ equals the solution to a convenient linear program:

$$\begin{aligned} \underline{\theta}_{\text{IV}}(c^S) = \quad \min_{\xi \in \Xi} \sum_{y_0, y \in \mathcal{Y}, z \in \mathcal{Z}, x \in \mathcal{X}} & \omega(y, x)\; 1\{y_0 \neq y\}\xi(y_0, y, z \,|\, x) \qquad \text{subject to:} \\ (i) \quad \sum_{y_0 \in \mathcal{Y}} \xi(y_0, y, z \,|\, x) \;&=\; \Pr(Y_i = y, Z_i = z \,|\, X_i = x) \qquad \forall\, y, z, x \\ (ii) \quad \sum_{y \in \mathcal{Y}} \xi(y_0, y, z \,|\, x) \;&=\; \Pr(Z_i = z \,|\, X_i = x) \sum_{y \in \mathcal{Y}, z' \in \mathcal{Z}} \xi(y_0, y, z' \,|\, x) \qquad \forall\, y_0, z, x, \end{aligned}$$

where Ξ is the set of probability mass functions defined over $\mathcal{Y}\times\mathcal{Y}\times\mathcal{Z}$ for each $x\in\mathcal{X}$. In Part II, I reformulate this linear program.

Part I

For a given $\xi\in\Xi$, let $\mathcal{F}^*_\xi$ denote the subset of $\mathcal{F}^*_{\text{IV}}$ that is consistent with ξ:

$$\mathcal{F}^*_\xi \equiv \left\{ f\in\mathcal{F}^*_{\text{IV}} \;:\; \sum_{y_1\in\mathcal{Y},s\in\{0,1\}} 1\{sy_1+(1-s)y_0=y\}f(y_0,y_1,s,z\,|\,x) \;=\; \xi(y_0,y,z\,|\,x) \quad \forall y_0,y,z,x \right\}.$$

Clearly, $\mathcal{F}^*_{\text{IV}} \;=\; \left\{f\in\mathcal{F}^*_\xi \;:\; \xi\in\Xi\right\} \;=\; \left\{f\in\mathcal{F}^*_\xi \;:\; \xi\in\Xi \text{ and } \mathcal{F}^*_\xi\neq\varnothing\right\}$, so that

$$\underline{\theta}_{\text{IV}}(c^S) \quad=\quad \min_{\xi\in\Xi}\left(\min_{f\in\mathcal{F}^*_\xi}\theta(f;c^S)\right) \quad \text{subject to} \quad \mathcal{F}^*_\xi\neq\varnothing.$$

I will now show that:

$$\mathcal{F}^*_\xi\neq\varnothing \iff \begin{cases} \sum_{y_0\in\mathcal{Y}}\xi(y_0,y,z|x) \;=\; \Pr(Y_i=y,Z_i=z\,|\,X_i=x) \quad \forall y\in\mathcal{Y},z\in\mathcal{Z},x\in\mathcal{X} \\ \sum_{y}\xi(y_0,y,z|x) \;=\; \Pr(Z_i=z\,|\,X_i=x)\sum_{y,z'}\xi(y_0,y,z'|x) \quad \forall y_0\in\mathcal{Y},z\in\mathcal{Z},x\in\mathcal{X}. \end{cases} \tag{A.4}$$

To prove sufficiency, suppose that $\mathcal{F}^*_{\hat\xi}$ is non-empty for some $\hat\xi\in\Xi$, and let $\hat f\in\mathcal{F}^*_{\hat\xi}$. It follows that, for all $y\in\mathcal{Y},z\in\mathcal{Z}$,

$$\begin{aligned} & \sum_{y_0\in\mathcal{Y}}\hat\xi(y_0,y,z|x) \\ = & \sum_{y_0,y_1\in\mathcal{Y},s\in\{0,1\}} 1\{sy_1+(1-s)y_0=y\}\hat f(y_0,y_1,s,z\,|\,x) \\ = & \Pr(Y_i=y,Z_i=z\,|\,X_i=x), \end{aligned} \tag{A.5}$$

where the first equality holds by the definition of $\mathcal{F}^*_{\hat\xi}$ and the fact that $\hat f\in\mathcal{F}^*_\xi$, and the second equality holds because $\hat f\in\mathcal{F}^*_{\text{IV}}$, so that $\hat f$ satisfies restriction (R_{OE}). On the other hand, for every $y_0 \in \mathcal{Y}$, $z \in \mathcal{Z}$,

$$\begin{aligned}
\sum_{y \in \mathcal{Y}} \hat{\xi}(y_0, y, z|x) &= \sum_{y_1 \in \mathcal{Y}, s \in \{0,1\}} \sum_{y \in \mathcal{Y}} 1\{sy_1 + (1-s)y_0 = y\} \hat{f}(y_0, y_1, s, z \,|\, x) \\
&= \sum_{y_1 \in \mathcal{Y}, s \in \{0,1\}} \hat{f}(y_0, y_1, s, z \,|\, x) \\
&= \Pr(Z_i = z \,|\, X_i = x) \sum_{y_1 \in \mathcal{Y}, s \in \{0,1\}, z' \in \mathcal{Z}} \hat{f}(y_0, y_1, s, z' \,|\, x) \\
&= \Pr(Z_i = z \,|\, X_i = x) \sum_{y \in \mathcal{Y}, z' \in \mathcal{Z}} \hat{\xi}(y_0, y, z'|x), \qquad \text{(A.6)}
\end{aligned}$$

where I used the definition of $\mathcal{F}^*_{\hat{\xi}}$ and the fact that $\hat{f} \in \mathcal{F}^*_{\hat{\xi}}$ in the first and last equalities, and the third equality holds because $\hat{f} \in \mathcal{F}^*_{\mathrm{IV}}$, so that $\hat{f}$ satisfies restriction ($\mathrm{R_{IV}}$).

To prove necessity, suppose that a given $\xi \in \Xi$ satisfies the restrictions in the right-hand-side of (A.16) and define f_ξ as:

$$f_\xi(y_0, y_1, s, z \,|\, x) = \begin{cases} 1\{y_0 \neq y_1\}\xi(y_0, y_1, z \,|\, x) & \text{if} \quad s = 1, \\ 1\{y_0 = y_1\}\xi(y_0, y_1, z \,|\, x) & \text{if} \quad s = 0. \end{cases} \qquad \text{(A.7)}$$

I will show that $f_\xi \in \mathcal{F}^*_\xi$. Notice first that $f_\xi \in \mathcal{F}$, since $\xi \in \Xi$. In addition, f_ξ satisfies ($\mathrm{R_{OE}}$): for every $y \in \mathcal{Y}$, $z \in \mathcal{Z}$ and $x \in \mathcal{X}$,

$$\begin{aligned}
&\sum_{y_0, y_1, s} 1\big\{sy_1 + (1-s)y_0 = y\big\} \; f_\xi(y_0, y_1, s, z \,|\, x) \\
&= \sum_{y_0} \sum_{y_1} \Big(1\big\{y_1 = y\big\} \; 1\{y_0 \neq y_1\} \; \xi\big(y_0, y_1, z|x\big) + 1\big\{y_0 = y\big\} \; 1\{y_0 = y_1\} \; \xi\big(y_0, y_1, z|x\big)\Big) \\
&= \sum_{y_0} \xi(y_0, y, z|x) \\
&= \Pr(Y_i = y,\; Z_i = z \,|\, X_i = x),
\end{aligned}$$

where the last equality is shown in (A.5). Finally, f_ξ satisfies ($\mathrm{R_{IV}}$): for all $y_0 \in \mathcal{Y}$ and

$z \in \mathcal{Z}$ and $x \in \mathcal{X}$,

$$
\begin{aligned}
\sum_{y\in\mathcal{Y}, s\in\{0,1\}} f_\xi(y_0, y, s, z \,|\, x) &= \sum_{y_1\in\mathcal{Y}, s\in\{0,1\}} \sum_{y\in\mathcal{Y}} 1\{sy_1 + (1-s)y_0 = y\} f_\xi(y_0, y_1, s, z \,|\, x) \\
&= \sum_{y\in\mathcal{Y}} \xi(y_0, y, z \,|\, x) \\
&= \Pr(Z_i = z \,|\, X_i = x) \sum_{y\in\mathcal{Y}, z'\in\mathcal{Z}} \xi(y_0, y, z' \,|\, x) \\
&= \Pr(Z_i = z \,|\, X_i = x) \sum_{y_1\in\mathcal{Y}, s\in\{0,1\}, z'\in\mathcal{Z}} f_\xi(y_0, y_1, s, z' \,|\, x),
\end{aligned}
$$

where the second and last equalities follow from the definition of $\mathcal{F}_\xi^*$, and the third equality is obtained from the steps shown in (A.6).

I now determine $\min_{f\in\mathcal{F}_\xi^*} \theta(f; c^S)$ for all non-empty $\mathcal{F}_\xi^*$. Consider a given ξ such that $\mathcal{F}_\xi^* \neq \varnothing$ and define f_ξ as in (A.7). I will show that $\theta(f_\xi; c^S) = \min_{f\in\mathcal{F}_\xi^*} \theta(f; c^S)$. The previous discussion showed that $f_\xi \in \mathcal{F}_\xi^*$. For any $f \in \mathcal{F}_\xi^*$, it follows that:

$$
\begin{aligned}
\theta(f_\xi; c^S) &= \sum_{y_0, y, s, z, x} \omega(y, x)\ 1\{s = 1\} f_\xi(y_0, y, s, z \,|\, x) \\
&= \sum_{y_0, y, z, x} \omega(y, x)\ f_\xi(y_0, y, 1, z \,|\, x) \\
&= \sum_{y_0, y, z, x} \omega(y, x)\ 1\{y_0 \neq y\} \xi(y_0, y, z \,|\, x) \\
&= \sum_{y_0, y, z, x} \omega(y, x)\ 1\{y_0 \neq y\} \sum_{y_1, s} 1\{sy_1 + (1-s)y_0 = y\} f(y_0, y_1, s, z \,|\, x) \\
&= \sum_{y_0, y, z, x} \omega(y, x)\ 1\{y_0 \neq y\} \sum_{y_1} 1\{y_1 = y\} f(y_0, y_1, 1, z \,|\, x) \\
&= \sum_{y_0, y, z, x} \omega(y, x)\ 1\{y_0 \neq y\} f(y_0, y, 1, z \,|\, x) \\
&\leq \sum_{y_0, y, z, x} \omega(y, x)\ f(y_0, y, 1, z \,|\, x) \\
&= \sum_{y_0, y, s, z, x} \omega(y, x)\ 1\{s = 1\} f(y_0, y, 1, z \,|\, x) \\
&= \theta(f; c^S),
\end{aligned}
$$

where the fourth equality follows from the fact that $f \in \mathcal{F}^*_\xi$. Hence,

$$
\begin{aligned}
\underline{\theta}_{\text{IV}}(c^S) \quad &\equiv \quad \min_{f \in \mathcal{F}^*_{\text{IV}}} \quad \theta(f;\, c^S) \\
&= \quad \min_{\xi \in \Xi} \left(\min_{f \in \mathcal{F}^*_\xi} \theta(f; c^S) \right) \quad \text{s.t.} \quad \mathcal{F}^*_\xi \neq \varnothing \\
&= \quad \min_{\xi \in \Xi} \sum_{y_0, y \in \mathcal{Y}, z \in \mathcal{Z}, x \in \mathcal{X}} \omega(y,x)\, 1\{y_0 \neq y\}\, \xi(y_0, y, z \,|\, x) \qquad \text{s.t.} \\
&(i) \quad \sum_{y_0 \in \mathcal{Y}} \xi(y_0, y, z \,|\, x) \;=\; \Pr(Y_i = y, Z_i = z \,|\, X_i = x) \qquad \forall\, y, z, x \\
&(ii) \quad \sum_{y \in \mathcal{Y}} \xi(y_0, y, z \,|\, x) \;=\; \Pr(Z_i = z \,|\, X_i = x) \sum_{y \in \mathcal{Y}, z' \in \mathcal{Z}} \xi(y_0, y, z' \,|\, x) \qquad \forall\, y_0, z, x
\end{aligned}
\tag{A.8}
$$

Part II

An equivalent formulation of (A.8) is:

$$
\begin{aligned}
&\min_{\phi \in \Phi, \psi \in \Psi} \sum_{z \in \mathcal{Z}, x \in \mathcal{X}} \Pr(Z_i = z \,|\, X_i = x) \sum_{y_0, y \in \mathcal{Y}} \omega(y,x)\, 1\{y_0 \neq y\}\, \psi(y_0, y \,|\, z, x) \qquad \text{s.t.} \\
&\quad (i) \quad \sum_{y_0 \in \mathcal{Y}} \psi(y_0, y \,|\, z, x) \;=\; \Pr(Y_i = y \,|\, Z_i = z, X_i = x) \qquad \forall\, y, z, x \\
&\quad (ii) \quad \sum_{y \in \mathcal{Y}} \psi(y_0, y \,|\, z, x) \;=\; \phi(y_0 \,|\, x) \qquad \forall\, y_0, z, x
\end{aligned}
\tag{A.9}
$$

where Φ is the set of probability mass functions defined over $\mathcal{Y}$ for each $x \in \mathcal{X}$ and Ψ is the set of probability mass functions defined over $\mathcal{Y} \times \mathcal{Y}$ for each $z \in \mathcal{Z}$ and $x \in \mathcal{X}$. To see that (A.8) and (A.9) are equal, let ξ^* solve problem (A.8) and define

$$
\begin{aligned}
\phi_{\xi^*}(y_0 \,|\, x) \quad &\equiv \quad \sum_{y \in \mathcal{Y}, z' \in \mathcal{Z}} \xi^*(y_0, y, z' \,|\, x) \qquad \forall y_0 \in \mathcal{Y},\ x \in \mathcal{X} \\
\psi_{\xi^*}(y_0, y \,|\, z, x) \quad &\equiv \quad \frac{\xi^*(y_0, y, z \,|\, x)}{\Pr(Z_i = z \,|\, X_i = x)} \qquad \forall y, y_0 \in \mathcal{Y},\ x \in \mathcal{X},\ z \in \mathcal{Z}.
\end{aligned}
$$

ϕ_{ξ^*} and ψ_{ξ^*} are well-defined conditional probability mass functions and they clearly yield the same objective. Moreover, ϕ_{ξ^*} and ψ_{ξ^*} are feasible in problem (A.9). Hence, (A.9) must be weakly smaller than (A.8). Conversely, let ϕ^* and ψ^* solve problem (A.9) and define

$$
\xi_{\psi^*}(y_0, y, z \,|\, x) \quad \equiv \quad \Pr(Z_i = z \,|\, X_i = x)\psi^*(y_0, y \,|\, z, x) \qquad \forall\, y, y_0 \in \mathcal{Y},\ x \in \mathcal{X},\ z \in \mathcal{Z}.
$$

ξ_{ψ^*} yields the same objective value and is well-defined: it is a probability mass function over $\mathcal{Y} \times \mathcal{Y} \times \mathcal{Z}$ for each $x \in \mathcal{X}$. Moreover, ξ_{ψ^*} is feasible in problem (A.8). Hence, (A.8) must be weakly smaller than (A.9). It follows that (A.8) and (A.9) are equal.

Finally, problem (A.9) equals

$$\min_{\phi \in \Phi} \sum_{z \in \mathcal{Z}, x \in \mathcal{X}} \Pr(Z_i = z \,|\, X_i = x)\; \lambda(\phi, z, x) \tag{A.10}$$

where

$$\begin{aligned} \lambda(\phi, z, x) \quad = \quad \min_{\gamma \in \Gamma} \sum_{y_0, y \in \mathcal{Y}} \omega(y, x)\; 1\{y_0 \neq y\}\; \gamma(y_0, y) \qquad \text{s.t.} \\ (i) \quad \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y) \;=\; \Pr(Y_i = y \,|\, Z_i = z, X_i = x) \qquad \forall\, y \\ (ii) \quad \sum_{y \in \mathcal{Y}} \gamma(y_0, y) \;=\; \phi(y_0 \,|\, x) \qquad \forall\, y_0 \end{aligned} \tag{A.11}$$

To see this, fix some $\phi \in \Phi$, and let $\gamma^*_{z,x}$ solve (A.11) for each $z \in \mathcal{Z}$ and $x \in \mathcal{X}$, given ϕ. Because ψ_{γ^*}, defined as $\psi(y_0, y \,|\, z, x) \equiv \gamma^*_{z,x}(y_0, y)$, is feasible in (A.9) and yields the same objective given ϕ, (A.9) is weakly smaller than (A.10). Similarly, let ψ^* solve the (A.9) given ϕ, and define $\gamma_{z,x}$ for all $z \in \mathcal{Z}$ and $x \in \mathcal{X}$ as $\gamma_{z,x}(y_0, y) \equiv \psi^*(y_0, y \,|\, z, x)$. For each (z, x), $\gamma_{z,x}$ is feasible in (A.11) and $\{\gamma_{z,x} : z \in \mathcal{Z}, x \in \mathcal{X}\}$ yield the same objective value. Therefore, (A.10) is weakly smaller than (A.9). It follows that problems (A.9) and (A.10) are equal. ■

Lemma 3. Let Assumption IVx hold and let Φ be the set of probability mass functions defined over $\{1, \ldots, n_Y\}$ for each $x \in \mathcal{X}$. Then, given $x^* \in \mathcal{X}$, $\Pr(S_i = 1 \,|\, X_i = x^*) \geq \mathrm{LB}(x^*)$ where

$$\mathrm{LB}(x^*) \;=\; \min_{\phi \in \Phi} \sum_{y \in \mathcal{Y}, z \in \mathcal{Z}} \frac{1}{2} \Pr(Z_i = z \,|\, X_i = x^*) \Big| \Pr(Y_i = y \,|\, X_i = x^*, Z_i = z) - \phi(y \,|\, x^*) \Big|,$$

and $\Pr(S_i = 1 \,|\, Y_i = y, X_i = x^*) \geq 0$. These lower bounds are sharp.

In addition, suppose that Z_i is binary, so that $\mathcal{Z} = \{0, 1\}$. Let $p_{\min} \equiv \min\big\{ \Pr(Z_i = 0 \,|\, X_i = x^*), \Pr(Z_i = 1 \,|\, X_i = x^*) \big\}$. Then

$$\mathrm{LB}(x^*) \;=\; \sum_{y \in \mathcal{Y}, z \in \mathcal{Z}} \frac{p_{\min}}{2} \Big| \Pr(Y_i = y \,|\, X_i = x^*, Z_i = 0) \;-\; \Pr(Y_i = y \,|\, X_i = x^*, Z_i = 1) \Big|$$

Proof. Fix $x^* \in \mathcal{X}$. Note that, under Assumption IVx, $\text{LB}(x^*) = \underline{\theta}_{\text{IV}}(c)$, where $c(y_0, y_1, s, z, x) = 1\{x = x^*\} \cdot 1\{s = 1\}$. By Lemma 2,

$$\underline{\theta}_{\text{IV}}(c) \quad = \quad \min_{\phi \in \Phi} \sum_{z \in \mathcal{Z}} \Pr(Z_i = z \,|\, X_i = x^*) \; \lambda(\phi, z, x^*) \tag{A.12}$$

where

$$\begin{aligned} \lambda(\phi, z, x^*) \; = \; \min_{\gamma \in \Gamma} \sum_{y_0, y \in \mathcal{Y}} & \; 1\{y_0 \neq y\} \; \gamma(y_0, y) \qquad \text{s.t.} \\ (i) \quad & \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y) \; = \; \Pr(Y_i = y \,|\, Z_i = z, X_i = x^*) \qquad \forall \; y \\ (ii) \quad & \sum_{y \in \mathcal{Y}} \gamma(y_0, y) \; = \; \phi(y_0 \,|\, x^*) \qquad \forall \; y_0 \end{aligned} \tag{A.13}$$

and Γ is the set of probability mass functions defined over $\mathcal{Y} \times \mathcal{Y}$. Problem (A.13) is a Monge-Kantorovich transportation (optimal transport) problem with binary costs. With this particular cost structure, it admits a closed-form solution, given by half of the absolute distance between the marginal distributions (see Propositions 4.7 and 4.2 of Levin and Peres (2017) for a recent textbook treatment), so that:

$$\lambda(\phi, z, x^*) \; = \; \frac{1}{2} \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = z, X_i = x^*) - \phi(y \,|\, x^*) \Big|$$

In conjunction with (A.12), this result gives the desired lower bound for $\Pr(S_i = 1 \,|\, X_i = x^*)$, $\text{LB}(x^*)$.

If Z_i is binary and $\mathcal{Z} = \{0, 1\}$, then for any $\phi \in \Phi$,

$$\begin{aligned} & \sum_{z \in \mathcal{Z}} \Pr(Z_i = z \,|\, X_i = x^*) \; \lambda(\phi, z, x^*) \\ = \quad & \Pr(Z_i = 0 \,|\, X_i = x^*) \; \lambda(\phi, 0, x^*) \; + \; \Pr(Z_i = 1 \,|\, X_i = x^*) \; \lambda(\phi, 1, x^*) \\ \geq \quad & p_{\min} \; \lambda(\phi, 0, x^*) \; + \; p_{\min} \; \lambda(\phi, 1, x^*) \\ = \quad & \frac{p_{\min}}{2} \cdot \Bigg(\sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 0, X_i = x^*) - \phi(y \,|\, x^*) \Big| \\ & \qquad + \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 1, X_i = x^*) - \phi(y \,|\, x^*) \Big| \Bigg) \\ \geq \quad & \frac{p_{\min}}{2} \cdot \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 0, X_i = x^*) - \Pr(Y_i = y \,|\, Z_i = 1, X_i = x^*) \Big|, \end{aligned}$$

where $p_{\min} \equiv \min\left\{\Pr(Z_i = 0 \,|\, X_i = x^*), \Pr(Z_i = 1 \,|\, X_i = x^*)\right\}$ and the last inequality follows from the triangle inequality, a property of the absolute (ℓ_1, Taxicab, Manhattan) distance. But this lower bound is achieved by $\phi^*(\cdot, x^*)$, defined as:

$$\phi^*(y \,|\, x^*) \;=\; \begin{cases} \Pr(Y_i = y \,|\, Z_i = 0, X_i = x^*) & \text{if} \quad p = \Pr(Z_i = 1 \,|\, X_i = x^*) \\ \\ \Pr(Y_i = y \,|\, Z_i = 1, X_i = x^*) & \text{if} \quad p = \Pr(Z_i = 0 \,|\, X_i = x^*) \end{cases}$$

Therefore,

$$\text{LB}(x^*) \quad = \quad \frac{p_{\min}}{2} \cdot \sum_{y=1}^{n_Y} \Big| \Pr(Y_i = y \,|\, Z_i = 0, X_i = x^*) - \Pr(Y_i = y \,|\, Z_i = 1, X_i = x^*) \Big|.$$

On the other hand, given $y^* \in \mathcal{Y}$ and $x^* \in \mathcal{X}$, the sharp lower bound for $\Pr(S_i = 1 \,|\, Y_i = y^*, X_i = x^*)$ is $\underline{\theta}_{\text{IV}}(c_y)$, where

$$c_y(y_0, y_1, s, z, x) = \frac{1\{x = x^*, y_1 = y^*\}}{\Pr(Y_i = y^* \,|\, X_i = x^*)} \cdot 1\{s = 1\}.$$

By Lemma 2,

$$\underline{\theta}_{\text{IV}}(c_y) \quad = \quad \min_{\phi \in \Phi} \sum_{z \in \mathcal{Z}} \Pr(Z_i = z \,|\, X_i = x^*) \; \lambda(\phi, z, x^*) \tag{A.14}$$

where

$$\begin{aligned} \lambda(\phi, z, x^*) \;&=\; \min_{\gamma \in \Gamma} \sum_{y_0 \in \mathcal{Y}} \frac{1\{y_0 \neq y^*\}}{\Pr(Y_i = y^* | X_i = x^*)} \, \gamma(y_0, y^*) \qquad \text{s.t.} \\ &(i) \quad \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y) \;=\; \Pr(Y_i = y \,|\, Z_i = z, X_i = x^*) \qquad \forall\, y \\ &(ii) \quad \sum_{y \in \mathcal{Y}} \gamma(y_0, y) \;=\; \phi(y_0 \,|\, x^*) \qquad \forall\, y_0 \end{aligned} \tag{A.15}$$

The constraints in problem (A.15) imply that

$$\begin{aligned} &\sum_{y_0 \in \mathcal{Y}} 1\{y_0 \neq y^*\} \gamma(y_0, y^*) \;-\; \sum_{y \in \mathcal{Y}} 1\{y \neq y^*\} \gamma(y^*, y) \\ &\qquad\qquad\qquad = \; \Pr(Y_i = y^* \,|\, Z_i = z, X_i = x^*) \;-\; \phi(y^* \,|\, x^*), \end{aligned}$$

so that the optimal solution to problem A.15 is given by any feasible γ^* such that:

$$\sum_{y_0 \in \mathcal{Y}} 1\{y_0 \neq y^*\} \gamma^*(y_0, y^*) \;=\; \max\Big\{0, \quad \Pr(Y_i = y^* \,|\, Z_i = z, X_i = x^*) \;-\; \phi(y^* \,|\, x^*)\Big\},$$

so that

$$\lambda(\phi, z, x^*) \;=\; \max\left\{0, \quad \frac{\Pr(Y_i = y^* \,|\, Z_i = z, X_i = x^*) \;-\; \phi(y^* \,|\, x^*)}{\Pr(Y_i = y^* \,|\, X_i = x^*)}\right\}.$$

Now, define $\phi^*(y \,|\, x^*) = \Pr(Y_i = y \,|\, Z_i = z^{\max}, X_i = x^*)$, where $z^{\max} = \arg\max_z \Pr(Y_i = y^* \,|\, Z_i = z, X_i = x^*)$. For all $z \in \mathcal{Z}$, it follows that

$$\Pr(Y_i = y^* \,|\, Z_i = z, X_i = x^*) \;-\; \phi(y^* \,|\, x^*) \leq 0.$$

Thus, $\lambda(\phi^*, z, x^*) = 0$ for all z, and $\underline{\theta}_{\text{IV}}(c_y) = 0$. ■

Lemma 4. Suppose that Z_i is degenerate and define $\mathcal{Y} \equiv \{1, \ldots, n_Y\}$. Consider the linear parameter $\theta(f; c^S)$ associated with scalar coefficients c^S, such that $c^S(y_0, y_1, s, z, x) \equiv \omega(y_1, x)1\{s = 1\}$ and $\omega : \mathcal{Y} \times \mathcal{X} \mapsto \mathbb{R}$. Then

$$\begin{aligned}
\underline{\theta}_{\text{PMF}}(c^S) \quad &\equiv \quad \min_{f \in \mathcal{F}^*_{\text{PMF}}} \quad \theta(f;\, c^S) \\
&= \quad \min_{\gamma \in \Gamma} \sum_{y_0, y \in \mathcal{Y}, x \in \mathcal{X}} \omega(y, x)\; 1\{y_0 \neq y\}\gamma(y_0, y \,|\, x) \qquad \text{subject to:} \\
&(i) \quad \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y \,|\, x) \;=\; \Pr(Y_i = y \,|\, X_i = x) \qquad \forall\, y, x \\
&(ii) \quad \sum_{y \in \mathcal{Y}} \gamma(y_0, y \,|\, x) \;=\; \Pr(Y_i(0) = y \,|\, X_i = x) \qquad \forall\, y_0, x
\end{aligned}$$

Proof. The proof is analogous to that of Lemma 2. In this proof, I simply show the main steps to avoid repetition.

For a given $\gamma \in \Gamma$, let $\mathcal{F}^*_\gamma$ denote the subset of $\mathcal{F}^*_{\text{PMF}}$ that is consistent with γ:

$$\mathcal{F}^*_\gamma \equiv \left\{ f \in \mathcal{F}^*_{\text{PMF}} \;:\; \sum_{y_1 \in \mathcal{Y}, s \in \{0,1\}} 1\{sy_1 + (1-s)y_0 = y\} f(y_0, y_1, s \,|\, x) \;=\; \gamma(y_0, y \,|\, x) \quad \forall y_0, y, x \right\}.$$

Thus,

$$\underline{\theta}_{\text{PMF}}(c^S) \quad = \quad \min_{\gamma \in \Gamma} \left(\min_{f \in \mathcal{F}^*_\gamma} \theta(f; c^S) \right) \quad \text{subject to} \quad \mathcal{F}^*_\gamma \neq \varnothing.$$

Moreover,

$$\mathcal{F}^*_\gamma \neq \varnothing \iff \begin{cases} \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y|x) \;=\; \Pr(Y_i = y \,|\, X_i = x) \quad \forall y \in \mathcal{Y}, x \in \mathcal{X} \\ \sum_{y} \gamma(y_0, y|x) \;=\; \Pr(Y_i(0) = y \,|\, X_i = x) \quad \forall y_0 \in \mathcal{Y}, x \in \mathcal{X}. \end{cases} \tag{A.16}$$

On the other hand,

$$\theta(f_\gamma;\, c^S) \quad = \quad \min_{f \in \mathcal{F}^*_\gamma} \theta(f; c^S),$$

where

$$f_\gamma(y_0, y_1, s \,|\, x) \quad = \quad \begin{cases} 1\{y_0 \neq y_1\}\gamma(y_0, y_1 \,|\, x) & \text{if} \quad s = 1, \\ \\ 1\{y_0 = y_1\}\gamma(y_0, y_1 \,|\, x) & \text{if} \quad s = 0. \end{cases} \tag{A.17}$$

We therefore conclude that:

$$\begin{aligned} \underline{\theta}_{\text{PMF}}(c^S) \quad &= \quad \min_{\gamma \in \Gamma} \sum_{y_0, y \in \mathcal{Y}, x \in \mathcal{X}} \omega(y, x)\; 1\{y_0 \neq y\}\gamma(y_0, y \,|\, x) \qquad \text{subject to:} \\ &(i) \quad \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y \,|\, x) \;=\; \Pr(Y_i = y \,|\, X_i = x) \qquad \forall\, y, x \\ &(ii) \quad \sum_{y \in \mathcal{Y}} \gamma(y_0, y \,|\, x) \;=\; \Pr(Y_i(0) = y \,|\, X_i = x) \qquad \forall\, y_0, x. \end{aligned}$$

■

Lemma 5. Let Assumption PMFx hold and suppose that Z_i is degenerate. Then, given $x^* \in \mathcal{X}$, the sharp lower bound for $\Pr(S_i = 1 \,|\, X_i = x^*)$ equals

$$\sum_{y \in \mathcal{Y}} \frac{1}{2} \Big| \Pr(Y_i = y \,|\, X_i = x^*) \;-\; \Pr(Y_i(0) = y \,|\, X_i = x^*) \Big|.$$

On the other hand, the sharp lower bound for $\Pr(S_i = 1 \,|\, Y_i = y, X_i = x^*)$ equals

$$\max \left\{ 0, \quad \frac{\Pr(Y_i = y^* \,|\, X_i = x^*) \;-\; \Pr(Y_i(0) = y^* \,|\, X_i = x^*)}{\Pr(Y_i = y^* \,|\, X_i = x^*)} \right\}.$$

Proof. Fix $x^* \in \mathcal{X}$. Note that, under Assumption PMFx, the sharp lower bound for $\Pr(S_i = 1 \,|\, X_i = x^*)$ is $\underline{\theta}_{\text{PMF}}(c)$, where $c(y_0, y_1, s, z, x) = 1\{x = x^*\} \cdot 1\{s = 1\}$. By Lemma 4,

$$\begin{aligned} \underline{\theta}_{\text{PMF}}(c) \quad &= \quad \min_{\gamma \in \Gamma} \sum_{y_0, y \in \mathcal{Y}} 1\{y_0 \neq y\}\; \gamma(y_0, y) \qquad \text{s.t.} \\ &(i) \quad \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y) \;=\; \Pr(Y_i = y \,|\, X_i = x^*) \qquad \forall\, y \\ &(ii) \quad \sum_{y \in \mathcal{Y}} \gamma(y_0, y) \;=\; \Pr(Y_i(0) = y_0 \,|\, X_i = x^*) \qquad \forall\, y_0 \end{aligned} \tag{A.18}$$

where Γ is the set of probability mass functions defined over $\mathcal{Y} \times \mathcal{Y}$. Like problem (A.13) in Lemma 3, Problem (A.18) is an optimal transport problem with binary costs and a well-known solution:

$$\underline{\theta}_{\text{PMF}}(c) \quad = \quad \sum_{y \in \mathcal{Y}} \frac{1}{2} \Big| \Pr(Y_i = y \,|\, X_i = x^*) \; - \; \Pr(Y_i(0) = y \,|\, X_i = x^*) \Big|.$$

On the other hand, given $y^* \in \mathcal{Y}$ and $x^* \in \mathcal{X}$, the sharp lower bound for $\Pr(S_i = 1 \,|\, Y_i = y^*, X_i = x^*)$ is $\underline{\theta}_{\text{PMF}}(c_y)$, where

$$c_y(y_0, y_1, s, z, x) = \frac{1\{x = x^*, y_1 = y^*\}}{\Pr(Y_i = y^* \,|\, X_i = x^*)} \cdot 1\{s = 1\}.$$

By Lemma 4,

$$\begin{aligned} \underline{\theta}_{\text{PMF}}(c_y) \quad = \quad & \min_{\gamma \in \Gamma} \sum_{y_0 \in \mathcal{Y}} \frac{1\{y_0 \neq y^*\}}{\Pr(Y_i = y^* | X_i = x^*)} \gamma(y_0, y^*) \qquad \text{s.t.} \\ & (i) \quad \sum_{y_0 \in \mathcal{Y}} \gamma(y_0, y) \; = \; \Pr(Y_i = y \,|\, X_i = x^*) \qquad \forall \, y \\ & (ii) \quad \sum_{y \in \mathcal{Y}} \gamma(y_0, y) \; = \; \Pr(Y_i(0) = y_0 \,|\, X_i = x^*) \qquad \forall \, y_0. \end{aligned}$$

Now, notice that the constraints in this problem imply that

$$\begin{aligned} & \sum_{y_0 \in \mathcal{Y}} 1\{y_0 \neq y^*\} \gamma(y_0, y^*) \; - \; \sum_{y \in \mathcal{Y}} 1\{y \neq y^*\} \gamma(y^*, y) \\ & \quad = \; \Pr(Y_i = y^* \,|\, X_i = x^*) \; - \; \Pr(Y_i(0) = y^* \,|\, X_i = x^*). \end{aligned}$$

Thus, the optimal objective is attained by any feasible γ^* that satisfies:

$$\sum_{y_0 \in \mathcal{Y}} 1\{y_0 \neq y^*\} \gamma^*(y_0, y^*) \; = \; \max \Big\{ 0, \quad \Pr(Y_i = y^* \,|\, X_i = x^*) - \Pr(Y_i(0) = y^* \,|\, X_i = x^*) \Big\},$$

so that

$$\underline{\theta}_{\text{PMF}}(c_y) \; = \; \max \left\{ 0, \quad \frac{\Pr(Y_i = y^* \,|\, X_i = x^*) - \Pr(Y_i(0) = y^* \,|\, X_i = x^*)}{\Pr(Y_i = y^* \,|\, X_i = x^*)} \right\}.$$

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